Written exam, 17 December 2003

YOUR ASSIGNMENT:

20 questions **Q1-Q20** are posed on the subsequent pages.

Q1-Q8, and **Q11-Q18** are *multiple choice questions*. For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation *clearly* to write, for example, "7B" as your answer to question Q7.

Q9-Q10 and **Q19-Q20** are ordinary *text questions*.

Each correct answer to a

- multiple choice question gives 4 points
- text question gives 9 points

The maximum score is thus 100 points.

Part I: "Tuberg" (Q1-Q10)

Part I deals with the daily delivery (on weekdays only) of beer from the Tuberg brewery to a number of *depots*. With each depot is associated a *sales district*, a geographical area within which a *depot manager* is in charge of the distribution of beer to his *customers* (supermarkets, kiosks, canteens, et cetera). As sales within a district may change over time, we shall furthermore consider problems of adjusting the borders between neighbouring districts, effected by moving customers from one district to another.

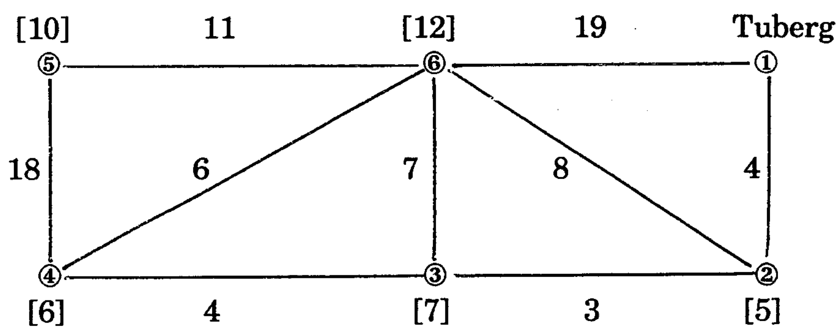

Fig. 1. The undirected network N = (V,U)

The vertex set V of the undirected network N=(V,U) consists of 6 vertices numbered as shown by the circled numbers. Vertex 1 represents the brewery and vertices 2,3,4,5,6 are the five depots. The edges of N (or the members of the edge set U) represent *road sections* with two connections to Tuberg and connections between each pair of *neighbouring* depots.

Numbers in brackets []: daily *demand d(j)* at depot j, j=2,3,4,5,6. Other numbers: *lengths c(i,j)*, all (i,j)∈U, of the corresponding edges, that is, the distance or travel time or their monetary equivalent between a pair (i,j) of vertices. All demands and lengths are integer-valued and assumed given "in appropriate units".

Big trucks are leaving Tuberg in the morning. Each truck goes directly to a depot and has sufficient capacity to supply all of its demand. The truck returns afterwards to Tuberg with empty bottles.

For all (i,j)∈U let f(i,j) be the nonnegative number of units of beer transported along edge (i,j) *from* i *to* j and let f(j,i) be analogously defined. Tuberg wishes to find a daily transportation plan satisfying all demands and minimising total cost z, that is

$$\text{minimise } z = \sum_{U} c(i,j)f(i,j)$$

The ensuing analysis is delegated to an *OR ana*lyst called *Orana*. The *QSB* (Quantitative Systems for Business, © Prentice-Hall, Inc.) LP-solver is at her disposal for the computations.

Since each edge of N is undirected and hence passable in either direction, the difference f(i,j)-f(j,i) can take any value, be it positive, zero, or negative. To simplify matters, Orana therefore introduces x(i,j) = f(i,j)-f(j,i) as a new decision variable for each edge (i,j)∈U. x(i,j) is thus a *free* variable, that is, unrestricted in sign.

For the x(i,j) variables to be well defined, all edges of U must be *directed*. Orana decides that edge (i,j), all (i,j)∈U, is directed *from* i *to* j if i<j. The result is the following *directed* network G=(V,E):

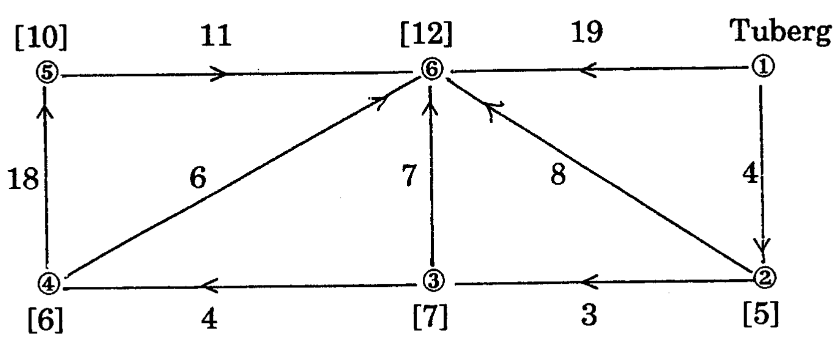

Fig. 2. The directed network G = (V,E)

Under the assumption that QSB can handle the free x(i,j) variables correctly, Orana designs the LP-model called **LP-P1** as {min z: Ax=b, x free} for the instance exhibited in fig. 2. Let *rank(A)* be the rank of the coefficient matrix A.

Q1: What is the value of rank(A)?

1A) 4 1B) 5 1C) 6 1D) 7

Let $S=(V,E_S)$ be the subgraph of G defined by the nonzero x(i,j) variables, that is

$$E_S = \{(i,j): (i,j)\in E \text{ and } x(i,j)\neq 0\}$$

A *shortest path tree* T rooted at vertex 1 is a spanning tree with the property that the unique path from vertex 1 to vertex j in T is a *shortest path* from vertex 1 to vertex j in G.

Q2: If **LP-P1** is capable of solving Tuberg's problem, how many of the following statements are true?

- Any basic feasible solution will have *exactly* 6 nonzero x(i,j)
- All x(i,j) in an optimal solution will become integer-valued
- E_S is a spanning tree
- E_S is a shortest path tree rooted at vertex 1

2A) 0   2B) 1   2C) 2
2D) 3   2E) 4

Q3: For the instance in fig. 2, QSB returns the following

x(1,2) = 40,   x(2,3) = 23,   x(2,6) = 12,
x(3,4) = 16,   x(4,5) = 10,   z = 569

Is it true that QSB actually has found an optimal solution to Tuberg's problem?

3A) yes   3B) no

Upon having identified the correct answer to **Q3** and under the assumption that QSB works as it should for any LP instance {min z: Ax=b, x≥0}, Orana draws the equally correct conclusion (which henceforth may be regarded as an established *fact*) that QSB considers *all variables to be nonnegative* even if this requirement is *not* explicitly made in the model.

Questions **Q4, Q6** deal both with the solution found by QSB in **Q3**. *Hint*: not as much as one single simplex iteration is needed to identify the correct answers!

Q4: For the four nonbasic variables, the QSB output shows the corresponding entries in the z row of the optimal simplex tableau:

Nonbasic variables	x(1,6)	x(3,6)	x(4,6)	x(5,6)
The z row	7	?	5	28

What is the value of the ?-entry under x(3,6)?

4A) 2   4B) 3   4C) 4   4D) 5   4E) 6

For each of the five depots represented by vertices in G, **LP-P1** must have a constraint, here, an *equation*, assuring that the demand is met.

Let **LP-D1** be the dual of **LP-P1** and let *w(j)* be the *dual variable* associated with vertex j, j=1, ... ,6.

The dual constraints tell us that

$$w(j) - w(i) \le c(i,j), \quad \text{all } (i,j) \in E$$

and, hence, that all dual variables are determined up to an arbitrary constant only. We can therefore fix one of them a priori and choose w(1)=0.

Q5: For the optimal solutions to the pair **LP-P1, LP-D1** of dual programmes, the *Complementary Slackness Conditions* assert that

5A) $$[x(i,j) + w(j) - w(i)]c(i,j) = 0, \text{ all } (i,j) \in E$$

and $$[d(j) + \sum_{(j,i)\in E} x(j,i) - \sum_{(i,j)\in E} x(i,j)]w(j) = 0, \quad \text{all } j \in V$$

5B) $$[c(i,j) + w(i) - w(j)]x(i,j) \ge 0, \text{ all } (i,j) \in E$$

and $$[d(j) + \sum_{(j,i)\in E} x(j,i) - \sum_{(i,j)\in E} x(i,j)]w(j) = 0, \quad \text{all } j \in V$$

5C) $$[c(i,j) + w(i) - w(j)]x(i,j) = 0, \text{ all } (i,j) \in E$$

5D) $$[d(j) + \sum_{(j,i)\in E} x(j,i) - \sum_{(i,j)\in E} x(i,j)]w(j) = 0, \quad \text{all } j \in V$$

Q6: Part of QSB output:

Vertices, j	1	2	3	4	5	6
w(j)	0	4	7	11	**?**	**?**

The missing two entries are

6A) 22,12 6B) 29,12 6C) 23,12
6D) 29,40 6E) 22,40

Temporary road repair on the section represented by edge (1,2) in G compels Tuberg to impose an *upper bound* u(1,2)=15 on x(1,2). Other changes in the road network necessitates a replacement of c(5,6)=11 by c(5,6)=11+Ω where Ω can take any integral value.

Orana modifies the model accordingly. A specific value of Ω is now part of the input. Let **LP-P2** be the result of her endeavours.

Q7: For **LP-P2** with Ω=14, QSB returns

7A) "Objective function value z = 579 (= 439+10×14)" ?

7B) "Unbounded solution" ?

7C) "No feasible solution" ?

Orana is pleased with the answer received in **Q7** prompting her to propose yet another modification of **LP-P2** by extending the edge set of G by the edge (6,2) with c(6,2)=8.

To summarize all the changes made, the new model called **LP-P3** is illustrated in fig. 3:

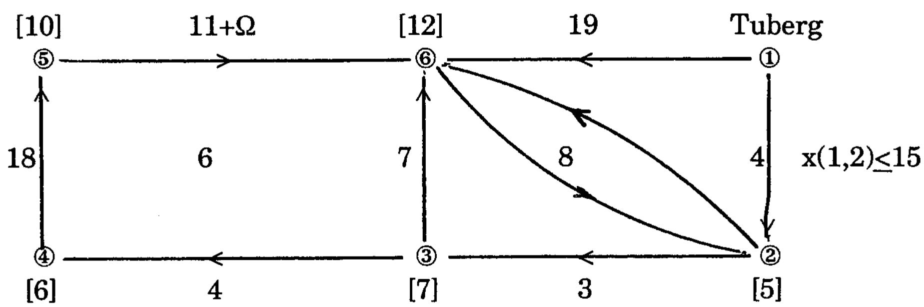

Fig. 3. **LP-3:** G is extended by the edge (6,2). $c(5,6)=11+\Omega$, $c(6,2)=8$, $x(1,2)\leq 15$

Q8: For **LP-P3** with Ω=0, QSB returns

x(1,2) = 15, x(1,6) = 25 x(2,3) = 23,
x(3,4) = 16, x(4,5) = 10, x(6,2) = 13, z = 952

Under the current conditions (road repair, upper bound on x(1,2), et cetera) for which values of Ω does **LP-P3** actually solve Tuberg's problem correctly

8A) $-8 \leq \Omega < 0$? 8B) $0 \leq \Omega < 18$? 8C) $18 \leq \Omega < 22$?

8D) $22 \leq \Omega$? 8E) no such value of Ω exists

For optimisation problems in general it is known that a *free* variable x_{27}, say, can be replaced by the difference between two *nonnegative* variables, that is,

x_{27} is free: $x_{27} = x'_{27} - x''_{27}$, $x'_{27} \geq 0$, $x''_{27} \geq 0$

Suppose that a certain model has φ free variables. If this kind of variable substitution is employed, the total number of variables will thus be *increased* by φ.

To economize with the number of variables, however, most textbooks recommend that only a *single* additional variable, here called *k*, is introduced. Thus, for example,

x_{27}, x_{39} are free: $x_{27} = x'_{27} - k$, $x_{39} = x'_{39} - k$, $x'_{27} \geq 0$, $x'_{39} \geq 0$, $k \geq 0$

Now, consider the following subgraph G* of G displayed in fig. 4 and note that c(1,2) has been changed to 30:

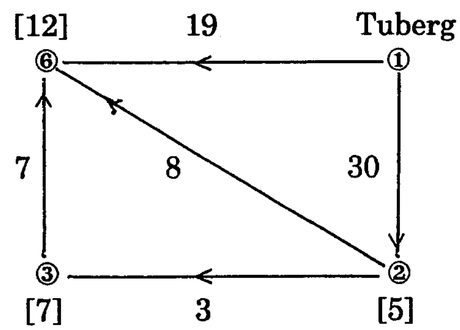

Fig. 4. The subgraph G* of G with c(1,2) = 30.

For each edge (i,j) of G* add the oppositely directed edge (j,i) with c(j,i) = c(i,j). Furthermore, replace the corresponding free variable x(i,j) by y(i,j) - k, y(i,j)$\geq$0, k$\geq$0.

This variable substitution will work for all values of k greater than or equal to a certain *threshold value k**.

Q9 *(text question)*:

What are the values of the y(i,j) variables and k* in an optimal solution to the instance shown in fig. 4?

We return to the *undirected* network N=(V,U), though with a few changes: the demands d(3) and d(5) are now 7+2Δ and 10+3Δ, respectively, where Δ is a nonnegative integer. Moreover, Tuberg itself has been removed. The new network is called N*=(V*,U*).

A depot manager's salary is partially based on the number of units of beer sold in his district, here represented by the demand. Sales within a district, however, may vary over time, for example, if a big shopping mall is being built somewhere and attract beer consumers from neighbouring districts.

It is Tuberg's policy that their depot managers should have fairly equal opportunities as reflected by their demands. It is therefore necessary from time to time to resize the districts so that they all have roughly the same demand.

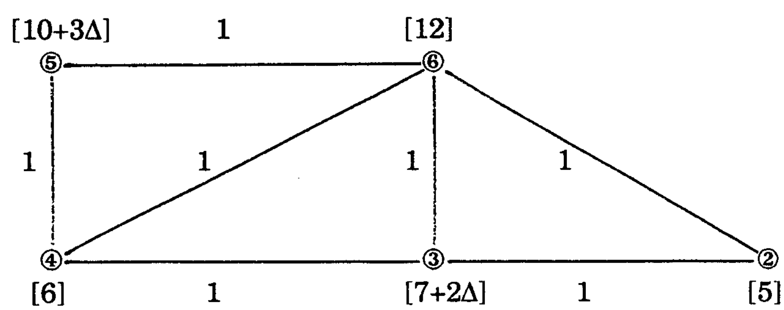

Fig. 5. The undirected network N* = (V*,U*). Total demand = 40+5Δ.
All edges have unity length.

Q10 *(text question):*

We note first that the total demand 40+5Δ is divisible by 5. We wish to redesign the 5 districts such that they all have a demand equalling 8+Δ. To this end, "units of demand" can be moved from one district (vertex) to another along the edges of N*. The *cost* of moving one unit from vertex i to vertex j is equal to the number of edges in a shortest path in N* between i and j.

Let m(i,j) be the number of units moved from i to j. Let d*(j), all j, be the desired demand at vertex j. For the instance shown in fig. 5, **LPP** is the LP-model for resizing the districts such that d*(j)=8+Δ, j=2,3,4,5,6, at a minimum total cost.

i) What does **LPP** look like?

ii) Solve **LPP** to optimality for Δ > 4. (*Hint:* still, detailed simplex iterations or the like are not needed).