# Written exam, 18 December 1997

## "The Wild Geese"

Maura and Sean is a middle-aged couple owning and running "The Wild Geese", a modest place in the beautiful Irish landscape west of *Caisleán an Bharraigh*. "The Wild Geese" houses a pub, a dining room, and a number of guest rooms. Being a capable cook, Sean is mainly in charge of food and beverages whereas Maura takes care of almost everything else.

Maura and Sean have recently celebrated their silver wedding. Among the gifts for the big event was a computer, some software packages and manuals. Day-to-day duties keep Maura and Sean fully occupied during the tourist season whereas less hectic periods allow for various leisure time activities. They have no experience whatsoever with computers but feel soon fascinated by their new tool's ability to play various card games. As time goes by, they also feel motivated to seek other challenges.

Among the software packages is one with the cryptic title: *Linear Programming and Extensions*. Some basic ideas are explained in the accompanying manual. Is it yet another game or is it conceivable that this package upon some preliminary exercises has something useful to offer as regards the daily management of "The Wild Geese"?

## YOUR ASSIGNMENT:

21 different questions Q1-Q21 are posed on the subsequent pages which also provide you with supplementary information when needed.

The last page is the form upon which the grading will be made with equal weights assigned to all questions. For each question, a number of possible answers is given. Tick the appropriate box and note that at most one box must be marked for each question since your answer to that question will otherwise be disregarded. Wrong answers are not penalized.

**Note:** This version of the original assignment covers **Q1-Q10** only. The last page with boxes to be ticked is **not** included.

## Questions Q1-Q4: Acceptable, luxury, and superb stocks

For a variety of meat dishes, a savoury *gravy* may give each dish its finishing touch. Savoury gravies, however, do not come from nothing. The foundation is a well-cooked *stock* prepared now and then in larger quantitities and kept for daily use in smaller portions in the freezer. Besides good basic ingredients (meat, bones, vegetables, spices, wine) the preparation of a stock requires a lot of time and patience. The blend must be boiled over a slow fire or *simmer* for hours and be reduced repeatedly until only the concentrate is left.

Sean, as we already know, is a capable cook but also anxious to further improve his skills. He met once with a French cook who indeed knew (from heritage?) how to make a perfect stock but was unwilling to reveal his secrets. Fortunately, Sean has come across a translation into English of a heavy volume authored by *Dr. Schnabelwasser*, a German nutrition expert who has earned a certain reputation (outside France) for his scientific approach towards the art of cooking.

Basically, a stock can be characterized by

|  |  |  |
|---|---|---|
|  | N: | the nutrition content, |
| and | F: | the flavour |

Some of Dr. Schnabelwassers's findings as regards the preparation of stocks are summarized in fig. 1. Each point in the plane represents a certain combination of N and F. For each point (N,F), the corresponding stock will be referred to as *S(N,F)*.

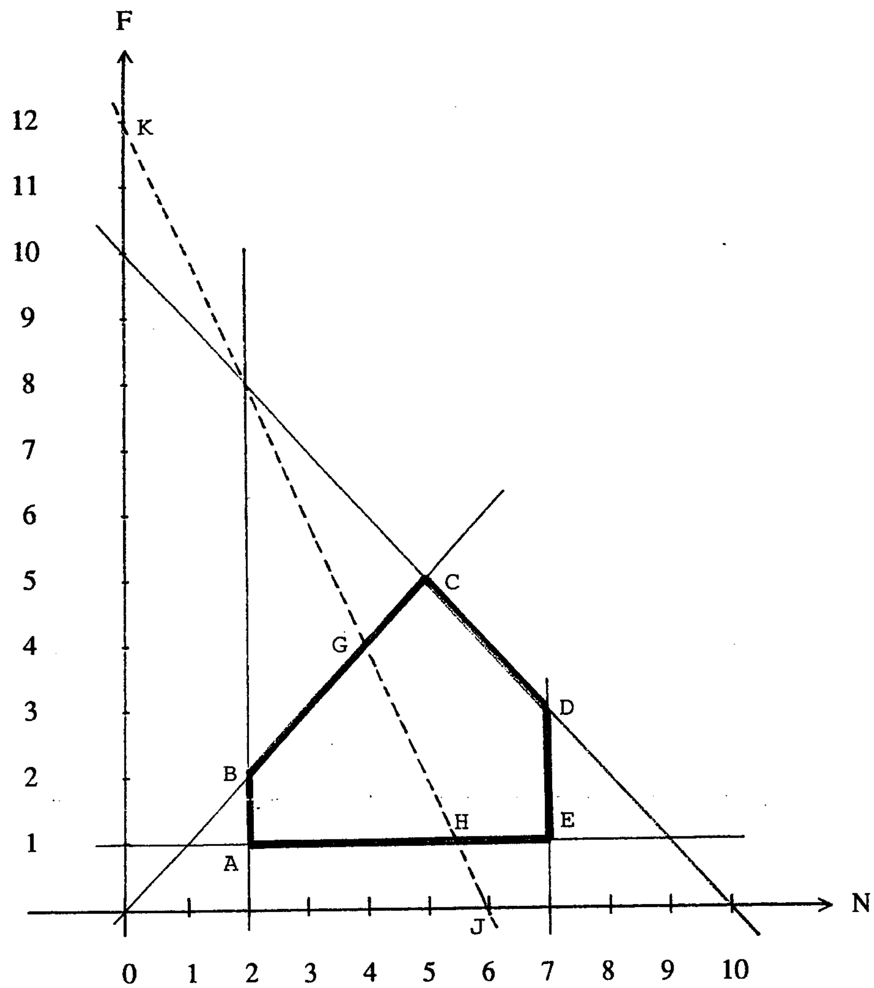


Fig. 1. A "map" of various stocks

A stock S(N,F) is called *acceptable* if (N,F) belongs to the pentagon ABCDE including the five lines defining it.

**Q1:** How can the pentagon ABCDE (or *feasible region* for a stock to be acceptable) be described in terms of N and F,

1a) $N \geq 2, N \leq 7, F \geq 1, N\text{-}F = 0, N+F = 10$

1b) $N \geq 2, N \leq 7, F \geq 1, N\text{-}F \leq 0, N+F \leq 10$

1c) $N \geq 2, N \leq 7, F \geq 1, N\text{-}F \geq 0, N+F \leq 10$

The dashed line in fig. 1 connects two points J,K with coordinates (6,0) and (0,12) respectively. G and H are the two points where the dashed line intersects the pentagon ABCDE. S(N,F) is called a *luxury* stock if (N,F) is a point within or on the smaller pentagon CDEHG. A luxury stock is thus an acceptable stock satisfying one additional constraint.

**Q2:** Which additional constraint must be satisfied by a luxury stock?

| | | | |
|---|---|---|---|
| 2a) | $2N+F \leq 12$, | 2b) | $2N+F \geq 12$, |
| 2c) | $N+2F \geq 12$, | 2d) | $2N-F \geq 12$ |

To distinguish between different stocks we need a measure to assess their *quality*. For a given stock S(N,F), let *Q(N,F) = 2N+F* be its quality. A *superb* stock is an *acceptable stock maximizing 2N+F.*

**Q3:** What is the quality of a superb stock?

| | | | | | |
|---|---|---|---|---|---|
| 3a) | 12 | 3b) | 13 | 3c) | 14 |
| 3d) | 15 | 3e) | 16 | 3f) | 17 |

For Q(N,F) = 2N+F it is seen that a superb stock is uniquely determined: there is one and only one feasible combination of N and F for which 2N+F attains its maximum value.

Suppose for a moment that Q(N,F) = 2N+F is replaced by Q(N,F) = *k*N+F where k is a constant.

**Q4:** For which value of k can we find at least two different stocks which both are superb?

| | | | |
|---|---|---|---|
| 4a) | k=½ | 4b) | k=¾ |
| 4c) | k=1 | 4d) | k=1½ |

## Questions Q5-Q10: Buying ingredients ...

Sean dreams of cooking a superb stock but his more price-conscious wife reminds him of the fact that "The Wild Geese" hardly ever will become famed as a gastronomic temple. Since Maura holds the purse strings, Sean agrees reluctantly to go for S(4,3) which is an acceptable and almost luxury stock, cf. fig. 1.

Cuts of beef or veal, a selection of vegetables, spices, wine, ...: not even Sean can prepare a proper stock from at most *five* ingredients only. To simplify matters, let us nevertheless assume that it can be done.

For each weight unit of the five ingredients considered, the following table lists the contents of N and F and the price in Irish pounds (IEP):

| Ingredient | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Content of N per weight unit | 1 | 0 | 2 | 2 | 1 |
| Content of F per weight unit | 0 | 1 | 3 | 1 | 1 |
| Price (IEP/ weight unit) | 4 | 7 | 24 | 12 | 10 |

For any combination of N and F it is assumed that the weight unit is defined such that the corresponding quantities of ingredients suffice for the preparation of one portion of S(N,F) of "desirable size".

Sean is told by Maura to buy ingredients at least total cost such that the amount of N and F is exactly 4 and 3 respectively. With decision variables $x_1, \dots, x_5$ representing the quantities (no. of weight units) of the five ingredients, the following LP-model is set up:

**LP1:** min $4x_1 + 7x_2 + 24x_3 + 12x_4 + 10x_5$ (total cost)

$1x_1 + 0x_2 + 2x_3 + 2x_4 + 1x_5 = 4$ (the N-constraint)

$0x_1 + 1x_2 + 3x_3 + 1x_4 + 1x_5 = 3$ (the F-constraint)

All variables nonnegative (nonnegativity)

The computer asserts that an optimal solution to LP1 is

$x_1 = 0, \quad x_2 = 0, \quad x_3 = ½, \quad x_4 = 1½, \quad x_5 = 0$

of total cost z = 30 IEP.

The solution is complemented by the information that the *shadow prices* $y_N$ and $y_F$ associated with the N- and the F-constraint respectively are

$y_N = 3$ and $y_F = 6$

**Q5:** Which one of the following statements is correct: the minimum total cost (IEP) for buying ingredients for the luxury stock S(5,4) is

5a) 30 + 3 ?
5b) 30 + 3 + 6 ?
5c) 30 + 6 ?
5d) 30 ?

A clever drug manufacturer realizes that the right contents of nutrition and flavour can be achieved by adding artificially produced N-pills and F-pills respectively to the stock and that Sean needs 4 N-pills and 3 F-pills for the preparation of one portion of S(4,3).

Let $y_N$, $y_F$ be the prices charged for each of these two kinds of pills. The drug manufacturer wants to maximize $4y_N + 3y_F$ which is the total price charged per portion. $y_N, y_F$, however, must be *competitive* in the sense that no cheaper solution exists for any combination of pills and ingredients.

To this end, an LP-model called **LP1-D** is set up by the drug manufacturer.

**Q6:** To reflect the above requirements correctly, what should LP1-D look like,

6a) max $\{ 4y_N + 3y_F: 4y_N + 3y_F \leq 30, y_N, y_F \geq 0 \}$
6b) max $\{ 4y_N + 3y_F: y_N \geq 4, y_F \geq 7, 2y_N+3y_F \geq 24, 2y_N+y_F \geq 12, y_N+y_F \leq 10,$ $y_N, y_F$ free $\}$
6c) max $\{ 4y_N + 3y_F: y_N \leq 4, y_F \leq 7, 2y_N+3y_F \leq 24, 2y_N+y_F \leq 12, y_N+y_F \leq 10 \}$
6d) Neither of the above

**Q7:** What is an optimal solution to LP1-D?

7a) $y_N = 3, y_F = 6$
7b) $y_N = 6, \quad y_F = 3$
7c) $y_N = 4, y_F = 5$
7d) $y_N = 4.5, y_F = 4$

Maura is willing to let Sean spend 30 IEP whenever a portion of S(4,3) is being prepared. It is seen that the quality Q(4,3) of S(4,3) equals 2×4+3 = 11. Now, is it conceivable that another stock would offer better "value for money", or, in other words, that Sean at a cost of at most 30 IEP can prepare another stock of a quality exceeding 11?

To investigate this question, the LP-model LP1 is replaced by

**LP2:** max $2N + F$ (maximize quality)

$$4x_1 + 7x_2 + 24x_3 + 12x_4 + 10x_5 \leq 30 \qquad \text{(total cost} \leq 30\text{)}$$

$$1x_1 + 0x_2 + 2x_3 + 2x_4 + 1x_5 = 4 \qquad \text{(the N-constraint)}$$
$$0x_1 + 1x_2 + 3x_3 + 1x_4 + 1x_5 = 3 \qquad \text{(the F-constraint)}$$

All variables nonnegative (nonnegativity)

**Q8:** What is your opinion about LP2 as it stands: which one of the following statements is incorrect?

8a) In an optimal solution to LP2, the x-variables will take the same values as in an optimal solution to LP1. Both N and F will approach infinity.

8b) LP2 is sheer nonsense.

8c) To avoid unbounded solutions, we need to impose the additional constraint $2N+F \leq L$ where L is a large number. LP2 will thereafter provide the correct answer to the question raised.

Both Maura and Sean have certain doubts as to model LP2's ability to come up with the right answer. Their next attempt is LP3 which includes both N and F in the constraints:

**LP3:** max $2N + F$

$$4x_1 + 7x_2 + 24x_3 + 12x_4 + 10x_5 \leq 30$$

$$1x_1 + 0x_2 + 2x_3 + 2x_4 + 1x_5 - N \geq 0$$
$$0x_1 + 1x_2 + 3x_3 + 1x_4 + 1x_5 - F \geq 0$$

All variables nonnegative

The computer: an optimal solution to LP3 is

$x_1 = 7.5$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$,
$N = 7.5$, $F = 0$

with Q(7.5, 0) = 15.

Sean is delighted with the high quality obtained whereas Maura is more sceptical and her scepticism, as we shall see, is well justified.

**Q9:** Which one of the following statements is incorrect?

9a) Dr. Schnabelwasser's quality measure is not appropriate for all combinations of (N,F).

9b) The computer has solved LP3 correctly.

9c) Future guests at "The Wild Geese" will surely praise gravies based on such a high-quality stock.

"Blood, toil, tears, and sweat": model building is an art! However, for LP3 extended by a single constraint, the computer's new bid for an optimal solution is

$x_1 = 4.5, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 1, \quad x_5 = 0, \qquad N = 6.5, F = 1$

with Q(6.5, 1) = 14.

Maura and Sean congratulate each other to their achievements: S(6.5, 1) is none less than a luxury stock which can be prepared for 30 IEP per portion.

**Q10:** Which one of the following constraints has been added to LP3?

10a) $N+F \leq 7.5$ 10b) $N+F = 7.5$

10c) $N-F \leq 7.5$ 10d) $F \geq 1$

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