Videregående Algoritmik (VA) <dikuex 052>

Written exam, 19 January 2005

YOUR ASSIGNMENT:

20 questions **Q1-Q20** are posed on the subsequent pages.

Q1-Q8 and **Q11-Q18** are *multiple choice questions*. For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation *clearly* to write, for example, "**7B**" as your answer to question Q7.

Q9-Q10 and **Q19-Q20** are ordinary *text questions*.

Each correct answer to a
- multiple choice question gives 4 points
- text question gives 9 points

The maximum score is thus 100 points.

Linear Programming (Q1-Q9)

Let $P(\alpha,\beta,\Delta)$ be the family of LP instances defined by the *standard form*

$$
\begin{array}{llllll}
\mathbf{P(\alpha,\beta,\Delta):} & \max & z = & 3x_1 + & \alpha x_2 & & \\
 & & & x_1 & & \le 4 & (C_1) \\
 & & & & (1/\beta)x_2 & \le 12 & (C_2) \\
 & & & 3x_1 + & 2x_2 & \le 6\Delta & (C_3) \\
 & & & x_1,\ x_2 \ge 0 & & &
\end{array}
$$

where α,β,Δ, $\alpha\ge 0$, $\beta>0$, $\Delta\ge 0$, are given constants to be viewed as *parameters*. Thus, for example, $P(5,½,3)$ is the instance obtained by substituting $5,½,3$ for α,β,Δ, respectively.

Let x_{j+2} be the slack variable to be inserted in constraint C_j, $j=1,2,3$. Convert $\mathbf{P(\alpha,\beta,\Delta)}$ into *slack form* which here must comprise *four* equations. Consider the *sum* of these four equations,

$$z + x_3 + x_4 + x_5 = (0 + 4 + 12 + 6\Delta) + \gamma_1 x_1 + \gamma_2 x_2$$

where γ_1 and γ_2 are constants.

Q1: What is the value of (γ_1,γ_2),

1A) $(1,\ \alpha-(1/\beta)-2)$ 1B) $(-1,\ \alpha-(1/\beta)-2)$

1C) $(-1,\ \alpha-(1/\beta)+2)$ 1D) $(-7,\ \alpha-(1/\beta)+2)$

For $\Delta=0$ the only *feasible* solution and also the only *basic feasible* solution is $x_1 = x_2 = x_5 = 0$, $x_3 = 4$, $x_4 = 12$. For other values of β,Δ, however, there may be several basic feasible solutions.

Q2: For given β (recall that $\beta>0$ by definition) and with Δ allowed to vary, how many basic feasible solutions *with* $x_1 > 0$ *and* $x_2 > 0$ can $\mathbf{P}(\alpha,\beta,\Delta)$ have,

2A) 1 only | 2B) 0 or 1 or 2

2C) 2 or 3 | 2D) at least 3

Q3: Let x_i,x_j,x_k be any subset of three variables selected among $x_1, \ldots, x_5$. Does there exist values of β,Δ for which $\mathbf{P}(\alpha,\beta,\Delta)$ has a basic feasible solution where all of x_i,x_j,x_k take positive values,

3A) yes | 3B) no

After two simplex iterations, the following slack form called **SF** is obtained:

SF:

$$
\begin{aligned}
z &= 6\Delta + 12\beta(\alpha-2) && + (2-\alpha)\beta x_4 && - x_5 \\
x_1 &= 2\Delta - 8\beta && + (2\beta/3)x_4 && - (1/3)x_5 \\
x_2 &= \mathbf{??} && - \beta x_4 && \\
x_3 &= 4 - (2\Delta - 8\beta) && - (2\beta/3)x_4 && + (1/3)x_5
\end{aligned}
$$

Q4: The value of x_2 appears to be missing. What should be substituted for **??**

4A) $2\Delta + \beta$ | 4B) 6Δ | 4C) 12β

4D) $2\Delta - 4\beta$ | 4E) $6\Delta + \beta$ | 4F) $(2-\alpha)\beta$

Q5: Assume $\alpha,\beta,\Delta > 0$. Which additional set of inequalities must be satisfied for **SF** to be an optimal solution to $\mathbf{P}(\alpha,\beta,\Delta)$,

5A) $\alpha \geq 2$, $0 \leq \Delta\beta - 24 \leq 12\beta$ | 5B) $0 \leq \Delta\beta - 24 \leq 12\beta$

5C) $\alpha \geq 2$, $0 \leq \Delta - 4\beta \leq 2$ | 5D) $\alpha \geq 2$, $0 \leq \Delta\beta - 24 \leq 2$

Let $\mathbf{D}(\alpha,\beta,\Delta)$ be the dual of $\mathbf{P}(\alpha,\beta,\Delta)$ and consider `INITIALIZE-SIMPLEX` with $\mathbf{D}(\alpha,\beta,\Delta)$ as input.

Q6: For which nonnegative values of α,β,Δ will `INITIALIZE-SIMPLEX` return "infeasible",

6A) $\Delta < 12(\alpha-2)$ | 6B) $\Delta > 12(\alpha-2)$

6C) $6\Delta > \beta$ | 6D) $\Delta\beta - 24 \leq 12\beta$

6E) no such values of α,β,Δ exist

Recall that a constraint is called *redundant* if its removal leaves the feasible region unaltered.

The feasible region is determined solely by the three constraints and the condition that all five variables are nonnegative.

Q7: For which values of β,Δ, $\beta>0$, $\Delta\geq 0$ will $\mathbf{P}(\alpha,\beta,\Delta)$ have no redundant constraints?

7A) $4\beta \leq \Delta \leq 2 + 4\beta$ 7B) $4\beta \leq \Delta \leq 2 + 4\beta$

7C) $2 < \Delta < 2 + 4\beta$ 7D) $\max \{2, 4\beta\} < \Delta < 2 + 4\beta$

For the slack form **SF**, assume that all basic variables are nonnegative but that the solution nevertheless is nonoptimal. Assume furthermore that $\mathbf{P}(\alpha,\beta,\Delta)$ has no redundant constraints.

Q8: In the next simplex iteration, x_4 is obviously the *entering* variable. Which one of the previous basic variables is *leaving*,

8A) x_1 8B) x_2 8C) x_3

Assume that **SF** exhibits an *optimal* solution to $\mathbf{P}(\alpha,\beta,\Delta)$ and that none of the three constraints C1-C3 is redundant. Let y_1^0, y_2^0, and y_3^0 be the dual variables in the corresponding optimal solution to $\mathbf{D}(\alpha,\beta,\Delta)$.

Q9 (text question):

Provide evidence to decide whether each of the following five statements is true or false,

9.1 $x_3 + x_4 > 0$ in any *feasible* solution to $\mathbf{P}(\alpha,\beta,\Delta)$

9.2 $x_3 + x_5 > 0$ in any *feasible* solution to $\mathbf{P}(\alpha,\beta,\Delta)$

9.3 An optimal *basic* solution to $\mathbf{D}(\alpha,\beta,\Delta)$ has at most two positive variables

9.4 $y_1^0 + y_2^0 + y_3^0 = 1 + (2-\alpha)\beta$

9.5 $y_1^0 + y_2^0 + y_3^0 = 1 + (\alpha-2)\beta$

Max flow (Q10)

Let $G_6 = (V,E)$ be a flow network with vertex set $V = (v_1, v_2, \ldots, v_6)$. The edge set E is defined by

$$E = \{(v_i,v_j): 1 \leq i < j \leq 6 \text{ and } i+j \text{ is odd}\}$$

The capacities $c(v_i,v_j)$ are determined by

$c(v_i,v_j) = 2j-i$: The capacity of edge (v_i,v_j), all $(v_i,v_j)\in E$.

Q10 (text question):

10.1 Find max flow in G_6 from v_1 to v_6. Identify the minimal cut.

10.2 Let G_{2n}, n = 1,2, ... , be defined analogously to G_6 above. Let v_1 and v_{2n} be the source and the sink, respectively.

Prove that that the number of edges in any flow augmenting path from v_1 to v_{2n} in G_{2n} is odd.

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