

# Semidefinite programming — an introduction

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## Exercise 1

Consider the positive semidefinite matrix:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \succeq 0$$

- show that  $a_{ii} \geq 0$
- show that if  $a_{ii} = 0$  then  $a_{ij} = 0$  for all  $j = 1, \dots, n$

## Exercise 2

A cone  $\mathcal{K}$  is an object satisfying:

$$\forall x \in \mathcal{K}, \lambda \geq 0 : \lambda x \in \mathcal{K}$$

- Prove that the set of positive semidefinite matrices is a cone.

For a given inner product  $\langle \cdot, \cdot \rangle$  the *dual cone* of  $\mathcal{K}$  is given by

$$\mathcal{K}^* = \{s \in \mathcal{S}_n^+ : \langle s, x \rangle \geq 0, \forall x \in \mathcal{K}\}$$

A cone  $\mathcal{K}$  is *self-dual* if an inner product can be chosen such that  $\mathcal{K} = \mathcal{K}^*$

- Show that the set of positive semidefinite matrices is a self-dual cone

## Exercise 3

The dense subgraph problem is defined as follows: Given a weighted graph  $G = (V, E, c)$  and a positive integer  $k$ . Choose a subset  $U \subseteq V$  of the nodes, with  $|U| = k$  such that the sum of the edge weights in the subgraph spanned by  $U$  is maximized. The problem can formally be defined as follows:

$$z = \max \left\{ \sum_{i \in U} \sum_{j \in U} c_{ij} : U \subseteq V, |U| = k \right\} \quad (1)$$

Formulate the dense subgraph problem as a semidefinite optimization problem. Consider various improved formulations.