Semidefinite Programming — an introduction

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Integer programming

$$\begin{array}{ll} \text{minimize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z} \end{array}$$

Solved through branch-and-bound, using lower bounds

- LP-relaxation can be solved efficiently
- Natural way of formulating ILP models

Many problems cannot be formulated as linear models, or the formulation is inefficient.

Interior point methods

Interior point methods can be extended to a number of cones (*self-dual homogeneous cones*)

- $\mathbb{R}^n$ (linear programming)
- vectorized symmetric matrices over real numbers (semidefinite programming)
- vectorized Hermitian matrices over complex numbers
- vectorized Hermitian matrices over quaternions
- vectorized Hermitian 3×3 matrices over octonions

Grötschel, Lovász and Schrijver [3]:
semidefinite programming solved in polynomial time

- *semidefinite relaxations* attractive tool
- Modeling part not well-developed

Concrete semidefinite optimizers: Sturm [6], Toh, Tutuncu and Todd [7], Fujisawa, Kojima and Nakata [2].

Semidefinite programming, definitions

- *vector* $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\qquad a^T = (a_1, \ldots, a_n)$
- *inner product between matrices* $A, B \in \mathbb{R}^{m,n}$
$$\langle A, B \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}.$$
- *vector product* of two vectors $a, b \in \mathbb{R}^n$ is
$$a^T b := \sum_{j=1}^{n} a_j b_j$$
$$ab^T := \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_n \\ \vdots & & \vdots \\ a_n b_1 & \cdots & a_n b_n \end{pmatrix}$$
- $\operatorname{diag}(A)$ *vector of diagonal elements* of $A \in \mathbb{R}^{n \times n}$
$$\operatorname{diag}(A) := \operatorname{diag} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{nn} \end{pmatrix}$$
- $\operatorname{Diag}(a)$ *diagonal matrix* of vector $a \in \mathbb{R}^n$
$$\operatorname{Diag}(a) = \operatorname{Diag}(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & a_n \end{pmatrix}$$

Semidefinite programming

- Matrix $A \in \mathbb{R}^{n\times n}$ is *positive semidefinite*

$$\forall y \in \mathbb{R}^n : y^T A y \geq 0 \tag{1}$$

- $\boxed{A \succeq 0} \Leftrightarrow A$ pos. semidef. and symmetric

Example

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \succeq 0$$

obviously symmetric
moreover $y^T A y \geq 0$ since

$$\begin{aligned} y^T A y &= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} y_1 + y_2 \\ y_1 + y_2 \end{pmatrix} \\ &= (y_1^2 + y_1 y_2) + (y_1 y_2 + y_2^2) \\ &= (y_1 + y_2)^2 \geq 0 \end{aligned}$$

Example

$$A = aa^T = \begin{pmatrix} a_1 a_1 & \cdots & a_1 a_n \\ \vdots & & \vdots \\ a_n a_1 & \cdots & a_n a_n \end{pmatrix}$$

is positive semidefinite (and symmetric) since

$$y^T A y = y^T (aa^T) y = (y^T a)(a^T y) = (a^T y)^2 \geq 0$$

Example

$$A = \mathrm{Diag}(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & a_n \end{pmatrix}$$

is positive semidefinite if and only if $a_i \geq 0$, $i = 1, \ldots, n$
to prove “if” note

$$y^T A y = \sum_{i=1}^{n} y_i^2 a_i \geq 0$$

to prove “only if” choose $y = (0, \ldots, 1, \ldots, 0)$

Cone

The set of semidefinite matrices is a cone (see exercise)

Properties of semidefinite matrices

Observations from linear programming:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \succeq 0$$

All square submatrices are again positive semidefinite

Diagonal elements $a_{ii} \geq 0$ (see exercise)

$$A_1 \succeq 0, A_2 \succeq 0, \ldots, A_k \succeq 0 \quad \Leftrightarrow \quad \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & A_k \end{pmatrix} \succeq 0$$

Characterisation of positive semiefinite matrices

Proposition 1 The following are equivalent

1 $A \in \mathbb{R}^{n\times n}$ is positive semidefinite

2 Eigenvalues $\lambda_i \geq 0$ for $i = 1, \ldots, n$

3 $\exists C \in \mathbb{R}^{n\times n}$ such that $A = C^T C$ and $\mathrm{rank}(C) = \mathrm{rank}(A)$

$A \in \mathbb{R}^{n\times n}$: eigenvector $v \neq 0$, eigenvalue λ $\quad \boxed{Av = \lambda v}$
$A = P \Lambda P^T$ eigenvalue decomposition of A

Proof

- **1** $\Rightarrow$ **2** Let $v \in \mathbb{R}^n$ be an eigenvector with $|v| = 1$ corresponding to λ. Since A is pos. semidef. $v^T A v \geq 0$ and $v^T (Av) = v^T \lambda v = \lambda v^T v = \lambda |v|^2 = \lambda$
- **2** $\Rightarrow$ **3** Let $A = P \Lambda P^T$ eigenvalue decomp. of A

$$\Lambda^{\frac{1}{2}} := \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$$

$$C := \Lambda^{\frac{1}{2}} P^T$$
$$C^T C = (P \Lambda^{\frac{1}{2}})(\Lambda^{\frac{1}{2}} P^T) = P \Lambda P^T = A$$

- **3** $\Rightarrow$ **1** Show: $\forall y \in \mathbb{R}^n : y^T A y \geq 0$
  Given $y \in \mathbb{R}^n$ let $w = Cy$

$$y^T A y = y^T (C^T C) y = (y^T C^T)(Cy) = w^T w \geq 0$$

Semidefinite programming

Linear optimization problem in standard form

$$\begin{array}{rl} \text{maximize} & cx \\ \text{subject to} & a_1 x = b_1, \\ & \vdots \\ & a_m x = b_m, \\ & x = (x_1, \ldots, x_n) \geq 0 \end{array}$$

Semidefinite optimization problem in standard form

$$\begin{array}{rl} \text{maximize} & \langle C, X \rangle \\ \text{subject to} & \langle A_1, X \rangle = b_1, \\ & \vdots \\ & \langle A_m, X \rangle = b_m, \\ & X \succeq 0 \end{array}$$

where $X = (X_{ij}) \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times n}$, $A_i \in \mathbb{R}^{n \times n}$, and $b_i \in \mathbb{R}$.
Grötschel, Lovász and Schrijver [3] polynomial algorithm

Several semidefinite matrices

Problem defined in several matrices

$$\begin{array}{rl} \text{maximize} & \sum_{j=1}^{k} \langle C_j, X_j \rangle \\ \text{subject to} & \sum_{j=1}^{k} \langle A_{1j}, X_j \rangle = b_1, \\ & \vdots \\ & \sum_{j=1}^{k} \langle A_{mj}, X_j \rangle = b_m, \\ & X_1 \succeq 0, \ldots, X_k \succeq 0 \end{array}$$

Remind that

$$X_1 \succeq 0, X_2 \succeq 0, \ldots, X_k \succeq 0 \quad \Leftrightarrow \quad \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & X_k \end{pmatrix} \succeq 0$$

Defining

$$X = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & X_k \end{pmatrix} \qquad A_i = \begin{pmatrix} A_{i1} & 0 & \cdots & 0 \\ 0 & A_{i2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & A_{ik} \end{pmatrix}$$

and noting that

$$\langle A_i, X \rangle = \left\langle \begin{pmatrix} A_{i1} & 0 & \cdots & 0 \\ 0 & A_{i2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & A_{ik} \end{pmatrix}, \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & X_k \end{pmatrix} \right\rangle = \sum_{j=1}^{k} \langle A_{ij} X_j \rangle$$

Semidefinite optimization problem in standard form

Embedding LP in semidefinite programming

Linear optimization problem

$$\begin{array}{rl} \text{minimize} & \sum_{j=1}^{k} c_j x_j \\ \text{subject to} & \sum_{j=1}^{k} a_{1j} x_j = b_1 \\ & \vdots \\ & \sum_{j=1}^{k} a_{mj} x_j = b_m \\ & x_1, \ldots, x_m \geq 0 \end{array}$$

Define matrices

- $A_{ij} = (a_{ij})$
- $X_i = (x_i)$

Nonnegativity constraint

$$X_i \succeq 0 \Leftrightarrow y X_i y \geq 0 \Leftrightarrow x_i y^2 \geq 0 \Leftrightarrow x_i \geq 0$$

We may use all LP-constraints as usual

Rank of matrices

$\text{rank}(A) \Leftrightarrow$ number of linearly independent columns of A

Proposition 2 matrix $X \in \mathbb{R}^{n \times n}$:

1. $X \succeq 0$ and $\text{rank}(X) = 1$

$\Leftrightarrow$

2. $X = xx^T$ for some vector $x \in \mathbb{R}^n$

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix} \qquad xx^T = \begin{pmatrix} x_1 x_1 & \cdots & x_1 x_n \\ \vdots & & \vdots \\ x_n x_1 & \cdots & x_n x_n \end{pmatrix}$$

Proof

- **1** $\Rightarrow$ **2** From Prop. 1: $X = C^T C$ where $C = \Lambda^{\frac{1}{2}} P^T$
  Since $\text{rank}(X) = 1$ only one eigenvalue $\lambda_i \neq 0$

  $$C = \Lambda^{\frac{1}{2}} P^T = \begin{pmatrix} 0 & & \\ & \sqrt{\lambda_i} & \\ & & 0 \end{pmatrix} \begin{pmatrix} p_{11} & \cdots & p_{n1} \\ \vdots & & \vdots \\ p_{1n} & \cdots & p_{nn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ c_1 & \cdots & c_n \\ 0 & \cdots & 0 \end{pmatrix}$$

  set $x = (c_1, \ldots, c_n)$. Then $X = C^T C = xx^T$.

- **2** $\Rightarrow$ **1** Construct matrix

  $$C := \begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ & & \\ 0 & \cdots & 0 \end{pmatrix} \qquad C^T C = \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ x_n & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ & & \\ 0 & \cdots & 0 \end{pmatrix} = xx^T$$

  Since $\text{rank}(C) = 1$ from Prop. 1: $X \succeq 0 \wedge \text{rank}(X) = 1$

Quadratic functions

If objective function or constraint

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_i x_j = x^T C x$$

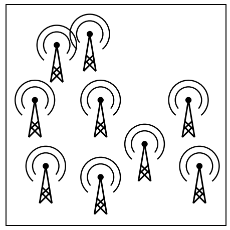
$\exists$ matrix $X = xx^T$ where $X \succeq 0$ and $\operatorname{rank}(X) = 1$ such that

$$\langle C, X\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}X_{ij}$$

since $X_{ij} = x_i x_j$ we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_i x_j = \langle C, X\rangle$$
$$X \succeq 0$$
$$\operatorname{rank}(X) = 1$$

The Quadratic Knapsack Problem



- $N = \{1, \dots, n\}$ items
- item $j \in N$ has weight w_j
- knapsack capacity c
- profit matrix $P = (p_{ij})$
- $p_{ij} + p_{ji}$ profit achieved if items i and j are selected

Binary variable $x_j = 1 \Leftrightarrow$ item j is selected

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in N}\sum_{j\in N} p_{ij}x_i x_j \\ \text{subject to} \quad & \sum_{j\in N} w_j x_j \le c, \\ & x_j \in \{0,1\}, \quad j \in N. \end{aligned} \tag{2}$$

Upper Bounds

LP-relaxation

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in N}\sum_{j\in N} p_{ij}x_i x_j \\ \text{subject to} \quad & \sum_{j\in N} w_j x_j \le c, \\ & 0 \le x_j \le 1, \quad j \in N. \end{aligned}$$

objective function is not linear

Bounds from Linearisation

Variables $y_{ij} = 1 \Leftrightarrow x_i = 1$ and $x_j = 1$

$$y_{ij} \le x_i, \qquad y_{ij} \le x_j, \qquad x_i + x_j \le 1 + y_{ij},$$

ILP model

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in N}\sum_{j\in N} p_{ij}y_{ij} \\ \text{subject to} \quad & \sum_{j\in N} w_j x_j \le c, \\ & y_{ij} \le x_i, && i,j \in N, \\ & y_{ij} \le x_j, && i,j \in N, \\ & x_i + x_j \le 1 + y_{ij}, && i,j \in N, \\ & x_j, y_{ij} \in \{0,1\}, && i,j \in N. \end{aligned}$$

LP-relaxation

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in N}\sum_{j\in N} p_{ij}y_{ij} \\ \text{subject to} \quad & \sum_{j\in N} w_j x_j \le c, \\ & y_{ij} \le x_i, && j \in N, \\ & y_{ij} \le x_j, && i \in N, \\ & x_i + x_j \le 1 + y_{ij}, && i,j \in N, \\ & 0 \le x_j \le 1, && j \in N, \\ & y_{ij} \ge 0, && i,j \in N. \end{aligned}$$

Upper bound U^1_{BC}

Bounds from Semidefinite Relaxation

Objective function of QKP

$$x^T P x = \sum_{i \in N} \sum_{j \in N} p_{ij} x_i x_j = \langle P, X \rangle$$

Formulation of QKP

$$\begin{array}{ll} \text{maximize} & \langle P, X \rangle \\ \text{subject to} & \langle \text{Diag}(w), X \rangle \leq c, \\ & X \succeq 0, \\ & \text{rank}(X) = 1, \\ & X_{ii} \in \{0,1\}. \end{array}$$

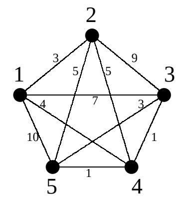
Dropping $\text{rank}(X) = 1$ constraint, relax last constraint

$$\begin{array}{ll} \text{maximize} & \langle P, X \rangle \\ \text{subject to} & \langle \text{Diag}(w), X \rangle \leq c, \\ & 0 \leq X_{ii} \leq 1 \\ & X \succeq 0, \end{array}$$

Upper bound U^0_{HRW}

Max-cut problem

Graph $G = (V, E, a)$, split $V = \{U, V \setminus U\}$



$$x_i = \begin{cases} 1 & \text{if } i \in U \\ -1 & \text{if } i \in V \setminus U \end{cases} \qquad \frac{1 - x_i x_j}{2} = \begin{cases} 1 & \text{if } x_i \neq x_j \\ 0 & \text{if } x_i = x_j \end{cases}$$

Model

$$\begin{array}{ll} \text{maximize} & \sum_{i<j} a_{ij} \frac{1 - x_i x_j}{2} \\ \text{subject to} & x \in \{-1, 1\}^n \end{array}$$

Rewriting objective

$$\begin{aligned} \frac{1}{2} \sum_{i<j} a_{ij}(1 - x_i x_j) &= \frac{1}{4} \sum_{i,j} a_{ij}(1 - x_i x_j) \\ &= \frac{1}{4} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_i x_i - \sum_{j=1}^{n} a_{ij} x_i x_j \right) \\ &= \frac{1}{4} x^T (\text{Diag}(Ae) - A) x \end{aligned}$$

Max-cut problem

Setting $C = \frac{1}{4}(\text{Diag}(Ae) - A)$ we get objective

$$\frac{1}{2} \sum_{i<j} a_{ij}(1 - x_i x_j) = x^T C x = \langle C, x x^T \rangle$$

Semidefinite program

$$\begin{array}{ll} \text{maximize} & \langle C, X \rangle \\ \text{subject to} & \text{diag}(X) = e \\ & X \succeq 0, \\ & \text{rank}(X) = 1, \end{array}$$

Since $\text{diag}(X) = e$ we have

$$X_{ii} = 1 = x_i x_i$$

Drop $\text{rank}(X) = 1$, semidefinite relaxation

$$\begin{array}{ll} \text{maximize} & \langle C, X \rangle \\ \text{subject to} & \text{diag}(X) = e \\ & X \succeq 0 \end{array}$$

Quadratic 0-1 programming

Problem

$$\max_{x \in \{0,1\}^n} x^T C x$$

Semidefinite program

$$\begin{array}{ll} \text{maximize} & \langle C, X \rangle \\ \text{subject to} & X \succeq 0, \\ & \text{rank}(X) = 1, \\ & X_{ii} \in \{0,1\} \end{array}$$

Drop $\text{rank}(X) = 1$, LP-relax $X_{ii} \in \{0,1\}$

Semidefinite relaxation

$$\begin{array}{ll} \text{maximize} & \langle P, X \rangle \\ \text{subject to} & 0 \leq X_{ii} \leq 1 \\ & X \succeq 0 \end{array}$$

Tighter bounds for QKP

Proposition 3 If $X \succeq 0$ and $\text{rank}(X) = 1$ and $X_{ii} \in \{0,1\}$ then also

$$X - \text{diag}(X)\,\text{diag}(X)^T \succeq 0. \qquad (3)$$

Proof

- $X = xx^T$ due to Proposition 2

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix} \qquad xx^T = \begin{pmatrix} x_1x_1 & \cdots & x_1x_n \\ \vdots & & \vdots \\ x_nx_1 & \cdots & x_nx_n \end{pmatrix}$$

- $\text{diag}(X) = \text{diag}(xx^T) = x$ when $x \in \{0,1\}^n$
- $\forall v \in \mathbb{R}^n$: $(x+v)(x+v)^T \succeq 0$
  choose $v = -\text{diag}(X)$

$$\begin{aligned}&(x+v)(x+v)^T = \\ &xx^T + vx^T + xv^T + vv^T = \\ &X + v\,\text{diag}(X)^T + \text{diag}(X)v^T + vv^T = \\ &X + (v + \text{diag}(X))(v + \text{diag}(X))^T - \text{diag}(X)\,\text{diag}(X)^T = \\ &X - \text{diag}(X)\,\text{diag}(X)^T \succeq 0\end{aligned}$$

Tighter bounds for QKP

Proposition 4

$$X - \text{diag}(X)\,\text{diag}(X)^T \succeq 0 \qquad \Leftrightarrow \qquad \overline{X} \succeq 0$$

where

$$\overline{X} := \begin{pmatrix} 1 & \text{diag}(X)^T \\ \text{diag}(X) & X \end{pmatrix}.$$

Proof

Define the regular matrix B as

$$B = \begin{pmatrix} 1 & -\text{diag}(X)^T \\ 0 & I \end{pmatrix},$$

and observe that

$$\begin{aligned} B^T\overline{X}B &= \begin{pmatrix} 1 & 0 \\ -\text{diag}(X) & I \end{pmatrix}\begin{pmatrix} 1 & \text{diag}(X)^T \\ \text{diag}(X) & X \end{pmatrix}\begin{pmatrix} 1 & -\text{diag}(X)^T \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & X - \text{diag}(X)\,\text{diag}(X)^T \end{pmatrix}. \end{aligned}$$

Using $\overline{y} = B^{-1}y$ we have

$$y^T\overline{X}y = y^T(B^{-1})^T B^T\overline{X}BB^{-1}y = \overline{y}^T B^T\overline{X}B\overline{y},$$

hence: $\overline{X} \succeq 0 \Leftrightarrow B^T\overline{X}B \succeq 0$

$$B^T\overline{X}B \succeq 0 \;\Leftrightarrow\; 1 \succeq 0 \text{ and } X - \text{diag}(X)\,\text{diag}(X)^T \succeq 0$$

Tighter bounds for QKP (bound 1)

Relax QKP to QKP'

$$\begin{aligned} \text{maximize} \quad & \langle P, X \rangle \\ \text{subject to} \quad & \langle \text{Diag}(w), X \rangle \le c, \\ & X - \text{diag}(X)\,\text{diag}(X)^T \succeq 0, \\ & \text{rank}(X) = 1, \\ & X_{ii} \in \{0,1\} \end{aligned}$$

Drop $\text{rank}(X) = 1$ and relax last constraint to $0 \le X_{ii} \le 1$

$$\begin{aligned} \text{maximize} \quad & \langle P, X \rangle \\ \text{subject to} \quad & \langle \text{Diag}(w), X \rangle \le c, \\ & X - \text{diag}(X)\,\text{diag}(X)^T \succeq 0, \\ & X_{ii} \le 1 \end{aligned}$$

upper bound U^1_{HRW}

Tighter bounds for QKP (bound 2)

- $w^Tx = x^Tw$
- $w^Tx \le c \Rightarrow w^Txx^Tw \le c^2$
- $w^Txx^Tw = \langle ww^T, xx^T \rangle$
- relax xx^T to X

relaxation

$$\begin{aligned} \text{maximize} \quad & \langle P, X \rangle \\ \text{subject to} \quad & \langle ww^T, X \rangle \le c^2, \\ & X - \text{diag}(X)\,\text{diag}(X)^T \succeq 0 \end{aligned}$$

upper bound U^2_{HRW}

Tighter bounds for QKP (bound 3)

The third semidefinite relaxation is based on the observation that $w^T x \le c$ can be multiplied by the real number $w^T x$ on both sides gives the constraint $(w^T x)^2 \le w^T xc$, leading to the inequality

$$
\begin{aligned}
0 \le w^T x(c - x^T w) &= w^T x \begin{pmatrix} 1, x^T \end{pmatrix} \begin{pmatrix} c \\ -w \end{pmatrix} \\
&= \begin{pmatrix} 0, w^T \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1, x^T \end{pmatrix} \begin{pmatrix} c \\ -w \end{pmatrix}.
\end{aligned} \qquad (4)
$$

Setting $X' = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1, x^T \end{pmatrix} = x'x'^T$ the right hand side expression in (4) can be written

$$
\left\langle \begin{pmatrix} c \\ -w \end{pmatrix} \begin{pmatrix} 0, w^T \end{pmatrix}, X' \right\rangle. \qquad (5)
$$

This leads to the following relaxation:

$$
\begin{aligned}
\text{maximize} \quad & \langle P, X \rangle \\
\text{subject to} \quad & \left\langle \begin{pmatrix} c \\ -w \end{pmatrix} \begin{pmatrix} 0, w^T \end{pmatrix}, X' \right\rangle \ge 0, \\
& X - \operatorname{diag}(X) \operatorname{diag}(X)^T \succeq 0.
\end{aligned} \qquad (6)
$$

Solving the above problem gives bound U^3_{HRW}.

Tighter bounds for QKP (bound 4)

The last relaxation is obtained by multiplying the capacity constraint with each of the variables x_i for $i \in N$ getting $x_i w^T x \le x_i c$. By writing the vector product explicitly we get the sum

$$
\sum_{j \in N} w_j x_i x_j \le x_i c. \qquad (7)
$$

By introducing in (7) X_{ij} for $x_i x_j$ and X_{ii} for x_i we get

$$
\sum_{j \in N} w_j X_{ij} \le X_{ii} c, \qquad (8)
$$

which leads to the following relaxation:

$$
\begin{aligned}
\text{maximize} \quad & \langle P, X \rangle \\
\text{subject to} \quad & \sum_{j \in N} w_j X_{ij} - X_{ii} c \le 0, \quad \text{for } i \in N, \\
& X - \operatorname{diag}(X) \operatorname{diag}(X)^T \succeq 0,
\end{aligned} \qquad (9)
$$

giving the bound U^4_{HRW}.

Strength of bounds for QKP

Helmberg, Rendl and Weismantel [4] together with Bauvin and Goemans [1] prove that

$$
U^1_{\text{HRW}} \ge U^2_{\text{HRW}} \ge U^3_{\text{HRW}} \ge U^4_{\text{HRW}}
$$

Computational Experiments

Randomly generated instances by Gallo, Hammer and Simeone

Let Δ be the *density* of an instance

- weight w_j is randomly distributed in $[1, 50]$ while
- profits $p_{ij} = p_{ji}$ are nonzero with probability Δ, and in this case randomly distributed in $[1, 100]$.
- capacity c is randomly distributed in $[50, \sum_{j=1}^{n} w_j]$

Implementation and experimental study was carried out by Rasmussen and Sandvik [5]

		U^1_{GHS}		U^2_{GHS}		U^3_{GHS}		U^4_{GHS}		$\overline{U}^4_{\text{GHS}}$	
Δ	n	time	dev	time	dev	time	dev	time	dev	time	dev
5	40	0.000	6.38	0.002	6.38	0.001	6.38	0.000	6.38	0.000	6.38
	60	0.000	17.61	0.000	17.61	0.001	17.61	0.002	17.61	0.001	17.49
	80	0.000	22.70	0.000	22.70	0.000	22.70	0.001	22.70	0.001	22.70
	100	0.000	13.84	0.002	13.84	0.003	13.84	0.001	13.84	0.000	13.84
	120	0.001	25.04	0.001	25.04	0.002	25.03	0.001	25.02	0.001	25.02
	140	0.000	40.17	0.000	40.17	0.006	39.22	0.002	39.09	0.003	38.99
	160	0.001	21.53	0.001	21.53	0.003	21.32	0.005	21.27	0.005	21.24
	180	0.002	21.69	0.002	21.69	0.008	21.69	0.002	21.69	0.002	21.69
	200	0.002	18.06	0.004	18.06	0.008	18.06	0.002	18.06	0.004	18.06
	avrg	0.001	20.78	0.001	20.78	0.004	20.65	0.002	20.63	0.002	20.60
25	40	0.000	35.22	0.001	34.00	0.002	27.82	0.001	27.33	0.000	26.45
	60	0.000	21.47	0.000	20.99	0.000	18.08	0.001	17.88	0.001	17.76
	80	0.000	16.83	0.001	16.78	0.001	14.89	0.003	14.81	0.001	14.74
	100	0.000	61.73	0.003	61.36	0.002	57.57	0.000	57.41	0.001	57.17
	120	0.000	31.08	0.001	30.64	0.002	27.30	0.001	27.24	0.004	27.18
	140	0.002	36.89	0.002	36.89	0.004	35.66	0.001	35.62	0.005	35.57
	160	0.000	18.54	0.001	18.54	0.006	17.96	0.002	17.94	0.006	17.91
	180	0.000	31.80	0.002	31.44	0.008	29.96	0.003	29.95	0.005	29.92
	200	0.003	49.02	0.003	49.02	0.009	42.59	0.010	42.50	0.006	42.31
	avrg	0.001	33.62	0.002	33.29	0.004	30.20	0.002	30.08	0.003	29.89
50	40	0.000	35.76	0.000	34.91	0.001	26.01	0.001	25.48	0.000	24.95
	60	0.000	36.11	0.000	34.17	0.000	27.61	0.001	27.35	0.002	26.98
	80	0.000	18.48	0.000	18.39	0.003	14.86	0.002	14.76	0.001	14.64
	100	0.001	52.13	0.001	46.00	0.003	34.87	0.001	34.70	0.003	34.42
	120	0.001	31.47	0.005	29.80	0.003	22.59	0.002	22.51	0.003	22.36
	140	0.000	43.17	0.002	43.14	0.003	38.05	0.006	37.97	0.003	37.84
	160	0.000	23.59	0.002	22.41	0.003	16.80	0.004	16.76	0.006	16.67
	180	0.003	30.88	0.007	28.91	0.005	21.90	0.005	21.87	0.007	21.79
	200	0.000	29.32	0.005	28.60	0.010	23.23	0.007	23.20	0.007	23.15
	avrg	0.001	33.43	0.002	31.81	0.003	25.10	0.003	24.96	0.004	24.75
75	40	0.000	35.10	0.000	28.80	0.000	18.48	0.000	18.00	0.000	17.55
	60	0.001	37.38	0.000	29.12	0.000	17.36	0.000	17.06	0.002	16.73
	80	0.000	26.77	0.001	23.04	0.002	15.37	0.003	15.27	0.000	15.12
	100	0.000	31.20	0.001	22.89	0.004	13.85	0.002	13.77	0.002	13.63
	120	0.000	33.20	0.002	21.57	0.002	11.59	0.003	11.53	0.003	11.39
	140	0.000	36.53	0.002	27.27	0.005	16.80	0.005	16.75	0.004	16.66
	160	0.000	25.79	0.002	22.07	0.004	16.13	0.007	16.11	0.007	16.05
	180	0.002	52.66	0.006	34.91	0.006	20.86	0.009	20.82	0.006	20.71
	200	0.001	51.06	0.004	26.00	0.007	13.30	0.009	13.26	0.009	13.19
	avrg	0.000	36.63	0.002	26.19	0.003	15.97	0.004	15.84	0.004	15.67
95	40	0.000	63.84	0.001	35.23	0.003	17.33	0.000	16.72	0.000	16.15
	60	0.000	61.20	0.000	35.03	0.002	16.16	0.001	15.86	0.003	15.51
	80	0.000	39.89	0.001	24.19	0.000	12.88	0.002	12.74	0.002	12.59
	100	0.002	49.27	0.002	27.00	0.001	13.50	0.006	13.39	0.000	13.28
	120	0.002	27.16	0.002	18.52	0.002	10.64	0.004	10.59	0.004	10.52
	140	0.002	28.02	0.004	19.79	0.006	11.69	0.003	11.65	0.005	11.58
	160	0.000	34.82	0.002	23.34	0.004	12.11	0.008	12.08	0.008	12.01
	180	0.002	31.71	0.004	20.00	0.006	11.27	0.008	11.24	0.003	11.18
	200	0.002	53.23	0.003	31.67	0.009	14.33	0.007	14.30	0.015	14.21
	avrg	0.001	43.24	0.002	26.08	0.004	13.32	0.004	13.17	0.004	13.00
100	40	0.001	23.67	0.001	15.24	0.001	9.64	0.001	9.32	0.000	9.15
	60	0.000	19.93	0.000	11.09	0.001	6.09	0.001	5.96	0.001	5.88
	80	0.000	40.13	0.000	21.15	0.001	10.72	0.003	10.58	0.001	10.45
	100	0.001	33.11	0.001	19.42	0.003	10.46	0.003	10.38	0.002	10.29
	120	0.001	37.67	0.004	22.86	0.002	11.33	0.003	11.27	0.005	11.19
	140	0.000	26.21	0.002	17.50	0.006	10.35	0.004	10.31	0.004	10.25
	160	0.000	38.14	0.001	19.63	0.007	9.68	0.006	9.64	0.005	9.59
	180	0.002	46.30	0.005	24.20	0.005	9.86	0.007	9.83	0.009	9.77
	200	0.003	40.76	0.004	22.01	0.009	10.09	0.010	10.07	0.008	10.01
	avrg	0.001	33.99	0.002	19.23	0.004	9.80	0.004	9.71	0.004	9.62
total avrg		0.001	33.62	0.002	26.23	0.004	19.18	0.003	19.06	0.003	18.92

Table 1: Bounds from upper planes (Intel Pentium III, 933 MHz).

Δ	n	U_{CHM} time	U_{CHM} dev	$\hat{U}^2_{\mathrm{MV}}$ time	$\hat{U}^2_{\mathrm{MV}}$ dev	$\hat{U}^2_{\mathrm{BFS}}$ time	$\hat{U}^2_{\mathrm{BFS}}$ dev
5	40	0.0	0.80	0.3	0.27	3.6	0.74
	60	0.0	0.65	0.6	0.37	9.6	0.49
	80	0.0	0.45	1.4	0.29	15.8	0.43
	100	0.0	0.32	2.0	0.16	24.8	0.31
	120	0.0	0.28	3.2	0.17	40.4	0.30
	140	0.0	0.56	2.2	0.53	75.6	0.70
	160	0.0	0.13	12.5	0.08	76.5	0.14
	180	0.0	0.14	7.3	0.11	95.0	0.17
	200	0.1	0.08	22.0	0.05	111.0	0.09
	avrg	0.0	0.38	5.7	0.22	50.2	0.37
25	40	0.0	3.07	0.6	2.38	3.7	1.65
	60	0.0	0.58	1.8	0.37	8.5	0.35
	80	0.0	1.02	6.5	0.77	16.1	0.65
	100	0.0	2.48	3.0	2.44	30.0	0.68
	120	0.1	0.57	4.9	0.53	38.6	0.26
	140	0.1	1.46	2.4	1.46	59.4	0.61
	160	0.2	0.50	64.7	0.47	62.2	0.16
	180	0.2	0.74	34.8	0.73	93.5	0.27
	200	0.3	1.47	20.3	1.47	122.3	0.36
	avrg	0.1	1.32	15.5	1.18	48.3	0.55
50	40	0.0	3.70	1.1	3.33	3.7	1.33
	60	0.0	2.85	3.1	2.70	8.4	0.67
	80	0.0	0.82	9.4	0.69	14.7	0.38
	100	0.1	3.61	15.4	3.57	24.7	0.50
	120	0.2	1.40	48.5	1.35	36.1	0.35
	140	0.3	1.33	28.4	1.32	47.3	0.19
	160	0.5	1.16	164.9	1.12	66.1	0.20
	180	0.6	1.57	27.9	1.57	84.8	0.21
	200	0.8	0.83	43.8	0.83	107.0	0.16
	avrg	0.3	1.92	38.1	1.83	43.7	0.44
75	40	0.0	3.92	3.3	3.36	3.7	1.58
	60	0.0	2.15	6.4	1.88	8.1	0.81
	80	0.1	1.52	2.0	1.52	13.5	0.23
	100	0.2	1.96	15.1	1.93	23.3	0.41
	120	0.3	1.89	195.5	1.80	31.2	0.49
	140	0.5	1.94	30.6	1.94	47.5	0.19
	160	0.7	1.24	506.8	1.21	63.1	0.18
	180	0.8	1.80	14.4	1.80	77.8	0.12
	200	3.7	2.03	17.9	2.03	102.5	0.17
	avrg	0.7	2.05	88.0	1.94	41.2	0.46
95	40	0.0	9.30	0.4	9.28	3.8	1.60
	60	0.0	4.21	0.6	4.21	8.4	0.68
	80	0.1	2.59	11.2	2.54	14.6	0.71
	100	0.2	1.63	13.1	1.62	21.0	0.46
	120	0.4	1.71	89.4	1.69	32.7	0.28
	140	0.5	1.82	113.2	1.81	41.5	0.22
	160	0.8	2.18	221.6	2.18	62.9	0.33
	180	1.0	2.06	15.1	2.06	80.8	0.32
	200	1.2	3.38	563.3	3.37	103.7	0.15
	avrg	0.5	3.21	114.2	3.20	41.0	0.53
100	40	0.0	4.22	2.0	4.08	3.6	1.55
	60	0.0	2.14	11.2	2.05	8.2	0.99
	80	0.1	2.22	20.4	2.20	14.0	0.53
	100	0.2	2.90	4.4	2.90	22.0	0.27
	120	0.4	2.23	20.0	2.22	33.1	0.38
	140	0.6	1.93	122.7	1.92	47.9	0.26
	160	0.8	2.25	978.7	2.21	60.0	0.31
	180	1.0	2.54	21.1	2.54	78.3	0.24
	200	1.4	1.66	79.8	1.66	103.1	0.50
	avrg	0.5	2.45	140.0	2.42	41.1	0.56
total	avrg	0.3	1.89	66.9	1.80	44.3	0.49

Δ	n	$\hat{U}^2_{\mathrm{CPT}}$ time	$\hat{U}^2_{\mathrm{CPT}}$ dev	U^2_{BC} time	U^2_{BC} dev
5	40	0.0	1.23	25.2	0.70
	60	0.1	1.41	160.0	0.45
	80	0.2	1.56		
	100	0.4	1.32		
	120	0.7	1.37		
	140	1.5	2.15		
	160	2.1	1.49		
	180	3.7	1.64		
	200	4.8	1.44		
	avrg	1.5	1.51	92.6	0.57
25	40	0.0	2.91	35.5	2.21
	60	0.1	0.92	326.4	0.38
	80	0.3	1.34		
	100	0.7	2.79		
	120	1.0	0.93		
	140	2.1	1.96		
	160	2.5	1.01		
	180	4.5	1.18		
	200	7.7	1.91		
	avrg	2.1	1.66	181.0	1.30
50	40	0.1	3.29	55.9	1.70
	60	0.2	2.11	699.3	0.84
	80	0.3	0.85		
	100	0.7	2.40		
	120	1.2	1.31		
	140	2.2	1.45		
	160	2.9	0.89		
	180	5.2	1.57		
	200	6.0	0.84		
	avrg	2.1	1.64	377.6	1.27
75	40	0.1	2.45	29.4	1.59
	60	0.1	1.39	116.4	0.80
	80	0.3	1.03	978.8	0.22
	100	0.5	1.21	2110.9	0.41
	120	0.8	0.73	2982.4	0.47
	140	1.7	1.11		
	160	2.5	0.84		
	180	3.6	0.78		
	200	4.2	0.78		
	avrg	1.5	1.15	1243.6	0.70
95	40	0.1	2.69	17.7	1.59
	60	0.1	1.25	102.2	0.68
	80	0.3	1.12	445.8	0.70
	100	0.4	0.91	1346.5	0.46
	120	0.7	0.66		
	140	1.3	0.65		
	160	2.1	0.63		
	180	3.2	0.71		
	200	4.7	0.54		
	avrg	1.4	1.02	478.0	0.86
100	40	0.1	1.89	24.8	1.55
	60	0.1	1.17	151.1	0.98
	80	0.3	0.74	397.7	0.51
	100	0.5	0.61	1990.1	0.27
	120	0.9	0.74		
	140	1.3	0.53		
	160	1.9	0.50		
	180	2.9	0.46		
	200	4.2	0.66		
	avrg	1.3	0.81	640.9	0.83
total	avrg	1.7	1.30	631.4	0.87

Table 2: Bounds based on Lagrangian relaxation (left table) and bounds from Linearisation (right table). (Intel Pentium III, 933 MHz).

Δ	n	U^0_{HRW} time	U^0_{HRW} dev	U^1_{HRW} time	U^1_{HRW} dev	U^2_{HRW} time	U^2_{HRW} dev	U^3_{HRW} time	U^3_{HRW} dev	U^4_{HRW} time	U^4_{HRW} dev
5	40	3.9	3.40	5.6	1.29	5.3	1.22	5.9	1.22	14.2	1.21
	60	9.7	7.64	12.4	2.70	13.5	1.84	14.2	1.83	75.8	1.83
	80	25.5	10.23	29.1	3.60	34.2	1.95	36.8	1.95	222.1	1.95
	100	54.3	6.82	68.1	2.24	83.5	1.12	90.6	1.12	579.5	1.12
	120	125.4	11.02	143.0	4.19	170.4	1.58	180.1	1.58	1053.8	1.58
	140	243.1	17.97	273.7	8.24	309.4	2.47	327.9	2.47	1810.4	2.47
	160	408.0	10.42	509.7	3.95	566.3	1.03	619.8	1.03	2410.6	1.03
	180	644.5	11.45	799.2	5.01	909.3	1.19	1014.2	1.19	3928.0	1.19
	200	904.3	9.76	1107.9	3.43	1345.5	0.69	1564.6	0.69		
	avrg	268.8	9.86	327.6	3.85	381.9	1.46	428.2	1.45	1261.8	1.55
25	40	3.8	17.36	4.2	10.44	4.7	2.99	5.4	2.97	17.8	2.95
	60	10.0	11.79	11.5	5.07	14.8	0.71	15.3	0.70	69.6	0.70
	80	24.4	9.78	28.7	4.96	38.2	0.87	42.1	0.86	255.5	0.85
	100	55.2	31.42	61.1	18.31	72.0	1.23	80.6	1.22	537.0	1.21
	120	120.3	16.75	138.6	8.77	185.1	0.49	204.2	0.48	1090.5	0.47
	140	240.7	21.13	274.3	11.59	340.3	0.49	350.7	0.49	1968.3	0.49
	160	373.4	11.62	457.7	5.78	626.3	0.25	679.3	0.25	3146.9	0.25
	180	583.3	18.92	664.0	10.00	888.5	0.31	929.8	0.30	4885.8	0.30
	200	916.2	24.76	1019.1	14.33	1306.5	0.34	1320.2	0.34		
	avrg	258.6	18.17	295.5	9.92	386.3	0.85	403.1	0.85	1496.4	0.90
50	40	3.8	20.46	4.1	12.23	5.1	1.85	5.7	1.83	17.3	1.82
	60	10.1	21.03	11.0	12.16	13.9	1.03	15.7	1.01	76.0	0.99
	80	23.0	11.22	27.5	5.86	39.6	0.48	39.4	0.48	247.7	0.48
	100	55.1	27.69	61.0	16.58	79.2	0.66	87.7	0.63	576.6	0.61
	120	121.9	17.60	136.7	10.02	193.2	0.40	206.4	0.39	1185.8	0.38
	140	240.2	25.51	265.6	14.00	396.7	0.23	397.2	0.23	2370.4	0.23
	160	397.0	13.32	449.8	7.54	679.1	0.23	750.9	0.21	3333.8	0.21
	180	627.1	17.62	704.2	10.11	1022.0	0.21	1113.6	0.20	4668.1	0.20
	200	855.6	17.21	947.6	9.39	1399.9	0.13	1576.1	0.13		
	avrg	259.3	19.07	289.7	10.88	425.4	0.58	465.9	0.57	1559.5	0.61
75	40	4.0	21.46	4.0	12.94	5.1	1.88	5.4	1.87	19.8	1.87
	60	9.5	20.72	10.8	11.82	14.3	1.01	16.4	0.98	72.8	0.97
	80	22.5	16.06	25.6	9.05	41.1	0.28	42.3	0.27	241.4	0.27
	100	53.9	16.96	61.0	10.15	89.8	0.47	94.0	0.47	628.0	0.46
	120	114.6	16.19	126.7	9.88	185.7	0.68	210.3	0.65	1254.4	0.64
	140	236.0	19.73	262.4	11.88	406.1	0.24	447.8	0.22	2168.5	0.21
	160	382.5	15.76	453.0	8.68	713.6	0.21	749.2	0.20	3191.2	0.20
	180	651.2	25.19	697.3	15.25	1064.5	0.35	1358.0	0.31		
	200	913.6	20.60	963.6	13.32	1382.2	0.44	2036.5	0.36		
	avrg	265.3	19.19	289.4	11.44	433.6	0.61	551.1	0.59	1082.3	0.66
95	40	3.7	38.28	3.9	25.05	4.9	1.80	5.3	1.78	17.3	1.78
	60	9.4	34.16	10.7	20.90	14.9	0.82	16.6	0.79	75.6	0.78
	80	23.2	22.26	26.2	13.48	40.8	0.78	43.8	0.76	256.5	0.75
	100	54.6	24.82	59.6	14.78	96.8	0.65	102.5	0.61	655.5	0.59
	120	117.1	16.32	131.7	9.32	228.7	0.34	233.0	0.32	1248.6	0.31
	140	236.5	17.29	259.9	9.82	428.0	0.27	464.6	0.27	2165.8	0.27
	160	369.1	21.00	419.3	12.44	729.5	0.34	732.6	0.34	3426.4	0.34
	180	619.4	18.34	662.9	11.02	1182.8	0.37	1227.4	0.36	5121.6	0.35
	200	906.3	29.32	967.7	18.20	1678.9	0.16	1762.3	0.16		
	avrg	259.9	24.64	282.4	15.00	489.5	0.61	509.8	0.60	1620.9	0.65
100	40	3.7	17.04	4.1	10.59	5.8	1.66	5.9	1.66	21.1	1.66
	60	9.3	12.07	10.5	7.22	16.2	1.05	15.9	1.05	83.8	1.04
	80	22.3	22.30	25.8	13.17	38.1	0.59	41.5	0.58	258.0	0.57
	100	50.5	19.85	57.3	12.25	93.8	0.30	100.5	0.29	639.7	0.29
	120	119.5	22.26	130.4	13.04	238.6	0.40	235.2	0.40	1257.3	0.40
	140	238.9	16.86	260.2	9.61	459.0	0.27	457.8	0.26	2203.8	0.26
	160	405.8	20.41	442.0	12.41	717.2	0.39	803.8	0.38	3503.7	0.37
	180	620.9	24.25	678.2	14.85	1133.7	0.27	1210.9	0.26	4833.8	0.25
	200	893.2	21.91	970.0	13.16	1620.9	0.54	1757.8	0.53		
	avrg	262.7	19.66	286.5	11.81	480.4	0.61	514.4	0.60	1600.1	0.60
total	avrg	262.4	18.43	295.2	10.48	432.8	0.79	478.7	0.78	1444.4	0.83

Table 3: Bounds from Semidefinite Programming. (Intel Pentium III, 933 MHz).

Approximation algorithms

- Should run in polynomial time
- Bounds from semidefinite programming
- Semidefinite optimization can be solved in polynomial time

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