Chvatal Rank 1 Cuts and Combinatorial Bender's Cuts

Course: Recent Research Results

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Overview

- Valid Inequalities - Chvatal Rank 1 Cuts
- Reformulations - Combinatorial Bender's Cuts

Chvatal-Gomory Cuts

See Fischetti and Lodi [3]

Consider the MIP

$$\min\{cx : Ax \leq b, x \geq 0, x \text{integer}\}$$

and the polyhedra:

$$P := \{x \in \mathrm{R}_+ : Ax \leq b\}$$
$$P_I := conv\{x \in mathrmZ_+ : Ax \leq b\} = conv\{P \cap \mathrm{Z}\}$$

where A is an $n \times m$ matrix and A and b is integer

A Chvatal-Gomory (CG) cut valid for P_I is:

$$\lfloor uA \rfloor x \leq \lfloor ub \rfloor, \quad u \in \mathrm{R}_+$$

Rank

The rank h is given as:

$h = 0$ Original constraints in problem

$h = 1$ CG cuts with respect to $Ax \leq b$

$h \geq 2$ CG cuts with respect to $Ax \leq b$ and other CG cuts

The first Chvatal closure of P is:

$$P_1 := \{x \geq 0 : Ax \leq b, \lfloor uA \rfloor x \leq \lfloor ub \rfloor \forall u \in \mathrm{R}_+\}$$

Clearly

$$P_I \subseteq P_1 \subseteq P$$

But more interesting is

$$P_1 \subset P, \quad P_I \neq P$$

That is, P_1 is a better approximation of P_I than P

Sometimes $P_1 = P_I$, e.g. matching problems (Edmonds blossom inequalities)

Separation Problem

Chvatal Rank 1 cuts are NP-hard to separate

Separation Problem:
Given an $x^* \geq 0$ find a CG cut $\alpha x \leq \alpha_0$ which is maximally violated by x^*.

MIP model:

$$\begin{aligned} \max\ & \alpha x^* - \alpha_0 \\ \text{s.t. } & \alpha_j \leq uA_j && \text{for } j = 1, \dots, n \\ & \alpha_0 > ub - 1 \\ & u_i \geq 0 && \text{for } i = 1, \dots, m \\ & \alpha, \alpha_0 \text{ integer} \end{aligned}$$

where α_j and α_0 are the coefficients of $\lfloor uA_j \rfloor$ and $\lfloor ub \rfloor$

Reformulating Separation MIP

Observations:

- When $x_j^* = 0$ variable does not contribute to violation
- Optimal $\alpha_j = uA_j$
- Same holds for x^* at their upper bound, by complementing them
- Assume $u_i < 1$, since cuts with $u_i \geq 1$ are dominating

New MIP formulation:

$$\begin{aligned} \max \sum_{j \in J(x^*)} & \alpha_j x_j^* - \alpha_0 & \\ \text{s.t. } & f_j = uA_j - \alpha_j & j \in J(x^*) \\ & f_0 = ub - \alpha_0 & \\ & 0 \leq f_j \leq 1 - \delta & \text{for } j \in J(x^*) \cup \{0\} \\ & 0 \leq f_j \leq 1 - \delta & \text{for } i = 1, \ldots, n \\ & \alpha_j \text{ integer} & \text{for } j \in J(x^*) \cup \{0\} \end{aligned}$$

where $J(x^* := \{j \in \{1, \ldots, n\} : x_j^* > 0\}$

Slacks: $f_j = uA_j - \lfloor uA_j \rfloor$ in range $[0, 1 - \epsilon]$

Multiplier Selection

Making the cut sparser and stronger

- Many equivalent CG cuts exist
- Produce CG cuts with few non-zeroes
- Modify objective function to

$$\max \sum_{j \in J(x^*)} (\alpha_j x_j^* - \alpha_0) - \sum_{i=1}^{m} w_i u_i$$

with $w_i = 10e - 4$, i.e., a penalty term

Enhancing "collaborative behavior"

- Remove bounds on u variables
- Better performance of MIP heuristics in CPLEX

Test Results

MIPLIB 3.0

Density of CG cuts:

	v.best				v.dense: denser cuts				v.bound: $u_i \in [-0.99, 0.99]$			
ID	# iter.s	# cuts	CPU time	% gap closed	# iter.s	# cuts	CPU time	% gap closed	# iter.s	# cuts	CPU time	% gap closed
eil101	17	34	1.03	100.0	17	34	1.06	100.0	100	217	40.60	92.6
gr120	38	49	6.45	100.0	42	62	12.88	100.0	100	277	104.50	83.4
pr124	83	141	14.57	100.0	93	134	22.52	100.0	100	210	41.40	25.6
gr137	15	37	1.37	100.0	15	37	1.42	100.0	100	221	62.00	77.2
pr144	35	71	4.29	100.0	38	79	8.05	100.0	100	231	46.50	65.5
ch150	76	125	45.91	100.0	81	154	75.26	100.0	100	276	111.90	57.7
rat195	100	171	79.20	97.4	100	194	104.40	93.3	100	292	167.90	34.8
kroA200	26	81	6.21	100.0	26	81	6.34	100.0	100	250	87.50	80.3
kroB200	100	175	39.60	82.2	100	171	39.20	77.0	100	229	87.70	30.9
ts225	100	172	51.70	84.1	100	179	54.00	88.8	100	245	102.00	37.6
pr226	100	170	33.90	96.1	100	152	31.40	80.8	100	195	73.50	31.3
gr229	96	176	92.60	100.0	100	220	145.70	97.4	100	249	156.40	46.3
gil262	100	194	59.20	97.0	100	202	103.10	96.7	100	247	112.30	48.5
a280	89	129	39.47	100.0	70	100	33.84	100.0	100	176	102.00	50.0
lin318	100	169	73.30	68.0	100	183	93.90	69.4	100	273	210.20	26.6

Table 2. 2-Matching Problems: impact of the multiplier selection rules.

2-matching Problems

	ILOG-Cplex			cut-and-branch				
ID	% gap closed	nodes	time	# cuts	% gap closed	separation time	nodes	total time
kroB200	100.0	330,913	2,748.24	179	77.1	39.20	348	60.67
ts225	47.1	230,115	1h	193	85.9	46.10	1,018	75.39
pr226	55.0	288,901	1h	168	91.3	31.60	51	35.20
gr229	100.0	15,005	180.79	182	96.9	54.70	7	60.94
gil262	100.0	117,506	2,094.77	172	97.4	54.50	1	58.39
lin318	53.3	117,100	1h	182	69.5	62.60	24,734	814.74

Table 3. 2-Matching Problems: cut-and-branch approach.

Remarks on CG Rank 1 cuts

- Future direction for general MIP solving?
- Balas and Saxena [1] improving on these results
- Jepsen et al. [4] used a subset of the CG Rank 1 cuts in a Branch-and-Cut-and-Price context
- Long separation time – need for faster/better heuristics

Bender's Decomposition – revisited

See Martin [5]

- IP master problem
- LP sub-problem
- Fix last master problem solution in sub-problem and use duals to generate Bender's cut

Original Problem:

$$\begin{aligned} \min\ & cx + fy \\ \text{s.t. } & Ax + By \geq b \\ & y \in Y \\ & x \geq 0 \end{aligned}$$

Restricted Bender's Master Problem:

$$\begin{aligned} \min\ & z \\ \text{s.t. } & z \geq fy + \pi_i(b - By) && i \in \text{extreme points} \\ & 0 \geq \pi_i(b - By) && i \in \text{extreme rays} \\ & y \in Y \end{aligned}$$

Sub-problem:

$$\begin{aligned} \min\ & cx + f\bar{y} \\ \text{s.t. } & Ax \geq b - B\bar{y} && \text{(dual variable: } \pi) \\ & x \geq 0 \end{aligned}$$

Combinatorial Bender's Cuts

See Codato and Fischetti [2]

Motivation:

- big-M constraints are notoriously hard

Idea:

- Reformulate to avoid these constraints
- Use Bender's idea to derive combinatorial cuts based on big-M
- Idea is associated with finding a Minimal (or irreducible) Infeasible Subsystem (MIS)
- Polynomial separation time

Expectations:

- Tighter LP formulation
- Faster solution times on certain structured MIPs

Conditional constraints

Consider MIP:

$$\min\{cx : Fx \leq g, x \in \{0,1\}\}$$

plus *conditional* constraints

$$x_{j(i)} = 1 \Rightarrow a_i y \geq b_i, \quad \forall i \in I$$

and *unconditional* constraints

$$Dy \geq e$$

with the continuous variables y

Conditional constraints are often modeled as:

$$a_i y \geq b_i - M(1 - x_{j(i)}) \quad \forall i \in I$$

Conditional constraints can be:

- Time windows in routing problems
- Resource startup indicators in lot sizing problems

Combinatorial Bender's Cuts (CB)

Formulate CB as:

$$\sum_{i \in C} x_{j(i)} \leq |C| - 1$$

where $C \subseteq I$ induces a MIS

That is, a minimal set of rows such that

$$SLAVE(C) := \begin{cases} a_i y \geq b_i, & \forall i \in C \\ Dy \geq e \end{cases}$$

has no feasible solution

Separation Problem

Given $x^* \in [0,1]$ a CB is violated iff

$$\sum_{i \in C} (1 - x^*_{j(i)}) < 1$$

Hence, the separation problem is

1. Weigh each conditional constraint by $1 - x^*_{j(i)}$
2. Weigh each unconditional constraint by 0
3. Look for minimum-weight MIS

This is NP-hard

Polynomial time heuristic

1. Start with $C := \{i \in I : x^*_{j(i)} = 1\}$
2. Verify infeasibility with LP-tools
3. Make C minimal in a greedy way

Simple, but exact when x^* is integer.

The Problem

P is a MIP:

$$
\begin{aligned}
z^* = \min\ & cx + dy && && (1)\\
s.t.\ & Fx \leq g && && (2)\\
& Mx + Ay \geq b && && (3)\\
& Dy \geq e && && (4)\\
& x_j \in \{0, 1\} && \forall j \in B && (5)\\
& x_j \text{integer} && \forall j \in G && (6)
\end{aligned}
$$

The link between x and y is of type:

$$
m_{i,j(i)} x_{j(i)} + a_i y \geq b_i \qquad \forall i \in I \qquad (7)
$$

Master-slave setup

Situation: $d = 0$

Master:

$$z^* = \min cx + dy \tag{8}$$
$$s.t.\ Fx \leq g \tag{9}$$
$$x_j \in \{0, 1\} \qquad \forall j \in B \tag{10}$$
$$x_j \text{integer} \qquad \forall j \in G \tag{11}$$

Slave $SLAVE(\tilde{x})$ parameterized by $\tilde{x}$:

$$a_i y \geq b_i - M\tilde{x} \tag{12}$$
$$Dy \geq e \tag{13}$$

Adding CB cuts

Let x^* be an optimal integer solution to master

- If $SLAVE(x^*)$ is infeasible so is P
- Otherwise look for a MIS and add

$$\sum_{i \in C : x^*_{j(i)}=0} x_j + \sum_{i \in C : x^*_{j(i)}=1} (1 - x_j) \geq 1$$

  to master

CBs can be separated given any master solution

Separation procedure must be exact for integral solutions top prove optimality of P

The other case

Situation: $d \neq 0$ and $c = 0$

Modify problem as follows:

- Add
$$dy \leq UB - \epsilon$$
to the slave system
- If master solution x^* is feasible for the slave system, solve the LP
$$\min\{dy : Ay \geq b - Mx^*, Dy \geq e\}$$
and update incumbent
- P is optimal when master problem is infeasible

Situation: $d = 0$ and $c = 0$

- Cannot be handled by this method

Reformulation of MIPs

Constraint

$$m_{i,j(i)}x_{j(i)} + a_i y \geq b_i \qquad \forall i \in I \qquad (14)$$

can restrict applicability of method

- Introduce continuous copies x^c of x and link constraints

$$x_j^c = x_j, \quad \forall j \in B$$

- Let the master constraints be

$$Fx \leq g$$

  and let slave constraints be

$$\begin{aligned} &Mx^c + Ay \geq b \\ &Dy \geq e \\ &x = x^c \\ &cx^c + dy \leq UB - \epsilon \end{aligned}$$

Notes:

- Possible for all MIPs
- Move master constraints to slave by substituting x, Leaves only CBs in master

Test Results – 1

Solving Map Labeling and Statistical Analysis problems.

Problems solved to optimality:

Table 1. Problems solved to proven optimality by both Cplex and CBC.

		Subset 1				
		Execution times			Nodes	
File name	Opt.	Cplex h:m:s	CBC h:m:s	Ratio Cplex/CBC	Cplex	CBC
Statistical analysis						
Chorales-116	24	1:24:52	0:10:18	8.2	10,329,312	20,382
Balloons76	10	0:00:10	0:00:14	71.4	40,481	4
BCW-367	8	0:08:33	0:00:13	39.4	79,980	463
BCW-683	10	2:02:29	0:00:32	229.7	399,304	671
WPBC-194	5	0:57:17	0:03:32	16.2	806,188	26,439
Breast-Cancer-400	6	0:02:50	0:00:16	1,062	181,990	1
Glass-163	13	0:56:17	0:00:05	675.4	3,412,702	64
Horse-colic-151	5	0:04:50	0:00:23	12.6	135,018	2,184
Iris-150	18	0:09:29	0:01:10	8.1	970,659	1,290
Credit-300	8	0:19:35	0:00:02	587.5	176,956	66
Lymphography-142	5	0:00:11	0:00:01	11.0	8,157	106
Mech-analysis-107	7	0:00:05	0:00:01	5.0	11,101	68
Mech-analysis-137	18	0:07:44	0:00:27	17.2	938,088	1,888
Monks-tr-122	13	0:02:05	0:00:05	25.0	262,431	357
Pb-gr-txt-198	11	0:04:21	0:00:05	52.2	135,980	110
Pb-pict-txt-444	7	0:02:07	0:00:02	63.5	71,031	1,026
Pb-hl-pict-277	10	0:04:17	0:00:27	9.5	22,047	115
Postoperative-88	16	0:15:16	0:00:01	916.0	2,282,109	171
BV-OS-282	6	0:05:13	0:00:24	13.0	56,652	1,044
Opel-Saab-80	6	0:01:03	0:00:13	4.8	87,542	7,314
Bus-Van-437	6	0:09:17	0:00:28	19.9	55,224	6,795
HouseVotes84-435	6	0:04:59	0:00:11	27.2	42,928	734
Water-treat-206	4	0:01:10	0:00:06	11.7	12,860	482
Water-treat-213	5	0:17:00	0:00:51	20.0	168,656	4,036
Map labeling						
CMS 600-1	600	1:08:41	0:04:34	15.0	110,138	14
Total/Mean	—	8:29:51	0:24:11	21.1	20,797,534	64,373

Test Results – 2

Not solved to optimality

Table 3. Problems solved to proven optimality by neither code.

	Subset 3								
	Cplex				CBC				
File name	Best solution	Best bound	Gap (%)	Mem. MB	Best solution	Best bound	Gap (%)	Mem. MB	ΔGap (%)
Statistical analysis									
Flags-169	10	5.0	49.8	290	10	6.50	35.0	4,052	14.8
Horse-colic-253	13	5.0	61.5	279	13	8.91	31.5	3,394	30.0
Horse-colic-185	11	5.0	54.4	265	12	6.33	47.3	4,494	7.1
Solar-flare-1066	273	7.6	97.3	787	284	201.30	29.1	1,423	68.2
Total/Mean		—	65.5	1,621		—	35.7	13,363	30.0
Map labeling									
Berlin	37	47.8	29.1	1,063	38	43.0	13.1	1,952	16.0
CMS 900-0 (4S)	881	900	2.2	676	897	898.5	0.2	283	2.0
CMS 1000-0 (4S)	945	1,000	5.8	566	978	998.3	2.1	509	3.7
US-Abbrv	73	104.8	43.6	740	77	99.7	29.5	428	14.1
CMS 650-0 (4P)	611	650	6.4	764	633	646.9	2.2	1,658	4.2
CMS 650-1 (4P)	604	650	7.6	798	638	648.0	1.6	706	6.0
Total/Mean		—	15.8	3,261		—	8.12	5,536	7.7

Remarks on CB cuts

- Interesting approach to reformulate hard MIPs
- Does not work well with structure defining conditional constraints, e.g., time windows
- Can be used as cutting planes without reformulating original MIP

References

[1] E. Balas and A. Saxena. Optimizing over the split closure. Forthcoming in Mathematical Programming, 2005.

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[3] M. Fischetti and A. Lodi. *Integer Programming and Combinatorial Optimization - IPCO 2005, Lecture Notes in Computer Science 3509*, chapter Optimizing over the first Chvátal closure, pages 12–22. Springer-Verlag, Berlin Heidelberg, 2005.

[4] M. Jepsen, B. Petersen, S. Spoorendonk, and D. Pisinger. Subset row inequalities applied to the vehicle routing problem with time windows. Forthcoming in Operations Research, 2006.

[5] R. K. Martin. *Large Scale Linear and Integer Optimization.* Kluwer Academic Press, Bosten, USA, 1999.