Introduction to Optimization, DIKU 2005/06
Friday, January 6
David Pisinger

Program of the day:

- Relaxation strategies.
- Lagrangian relaxation.
- Example: A location problem solved through branch-and-bound, using Lagrangian relaxation

Relaxation

In a branch-and-bound algorithm we find upper bounds by relaxing the problem

Relaxation (Wolsey sec. 2.1)

$$\max\{cx : x \in S\} \qquad (IP)$$
$$\max\{f(x) : x \in T\} \qquad (RP)$$

RP is a relaxation of IP if

- $S \subseteq T$
- $f(x) \geq cx$ for all $x \in S$

Which constraints should be relaxed?

- Quality of bound (tightness of relaxation)
- Remaining problem can be solved efficiently
- Proper multipliers can be found efficiently
- Constraints difficult to formulate mathematically
- Constraints which are too expensive to write up

Overview

Different relaxations

- LP-relaxation
- Deleting constraint
- Lagrangian relaxation
- Surrogate relaxation
- Semidefinite relaxation

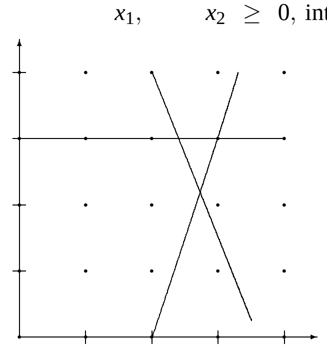
Relaxations are often used in combination.

Hierarchy

- Best surrogate relaxation
- Best Lagrangian relaxation
- LP-relaxation

Lagrangian relaxation, example

$$\begin{array}{llllll} \text{maximize} & 4x_1 & + & x_2 & & \\ \text{subject to} & 3x_1 & - & x_2 & \leq & 6 \\ & & & x_2 & \leq & 3 \\ & 5x_1 & + & 2x_2 & \leq & 18 \\ & x_1, & & x_2 & \geq & 0,\ \text{integer} \end{array}$$



IP solution $(x_1, x_2) = (2, 3)$ with $z_{IP} = 11$
LP solution $(x_1, x_2) = (\frac{30}{11}, \frac{24}{11})$ with $z_{LP} = \frac{144}{11} = 13.1$

Last constraint complicating, relax using multiplier $\lambda = \frac{1}{2}$

$$\begin{array}{llllll} \text{maximize} & 4x_1 & + & x_2 & - & \frac{1}{2}(5x_1 + 2x_2 - 18) = \frac{3}{2}x_1 + 9 \\ \text{subject to} & 3x_1 & - & x_2 & \leq & 6 \\ & & & x_2 & \leq & 3 \\ & x_1, & & x_2 & \geq & 0,\ \text{integer} \end{array}$$

Solution $(x_1, x_2) = (3, 3)$ with $z_{LR} = \frac{3}{2}3 + 9 = 13.5$
Upper bound

Lagrangian relaxation

Integer Programming Problem

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \le b \\ & Dx \le d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Lagrange relax $Dx \le d$, using multipliers $\lambda \ge 0$

$$\begin{array}{ll} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \le b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Proposition 1 Optimal solution to relaxed problem gives upper bound on original problem

Proof show that relaxation

multiplier λ_i "punishment"
If λ_i large $\Rightarrow$ constraint satisfied
If $\lambda_i = 0 \Rightarrow$ drop constrain

Lagrangian relaxation

Lagrange relaxed problem as function of $\lambda \ge 0$

$$\begin{array}{ll} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \le b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \ge 0} z_{LR}(\lambda)$$

Natural questions:

- How do we find best λ?
- How tight is relaxation?

Properties of Lagrange relaxation

Geom. interpretation, Lagrangian Relaxation

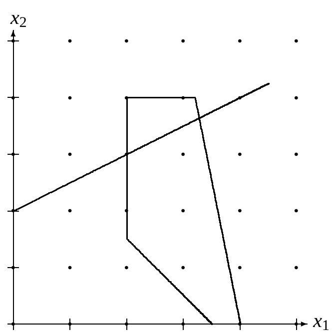
$$\begin{array}{lrcrcr} \max & 7x_1 & + & 2x_2 & & \\ \text{s.t.} & -x_1 & + & 2x_2 & \le & 4 \\ & 5x_1 & + & x_2 & \le & 20 \\ & -2x_1 & - & 2x_2 & \le & -7 \\ & -x_1 & & & \le & -2 \\ & & & x_2 & \le & 4 \\ & \multicolumn{5}{c}{x_1, x_2 \text{ integer}} \end{array}$$

First constraint "$-x_1 + 2x_2 \le 4$" is "complicating"
Lagrangian relax this constraint ($\lambda \ge 0$) getting objective

$$7x_1 + 2x_2 - \lambda(-x_1 + 2x_2 - 4)$$

Relaxed problem

$$\begin{array}{lrcrcr} \max & \multicolumn{5}{l}{(7+\lambda)x_1 + (2-2\lambda)x_2 + 4\lambda} \\ \text{s.t.} & 5x_1 & + & x_2 & \le & 20 \\ & -2x_1 & - & 2x_2 & \le & -7 \\ & -x_1 & & & \le & -2 \\ & & & x_2 & \le & 4 \\ & \multicolumn{5}{c}{x_1, x_2 \text{ integer}} \end{array}$$



Geom. interpretation, Lagrangian Relaxation

Original problem, integer solution

$(x_1, x_2) = (4, 0)$ $z = 28.00$

Original problem, LP-relaxed solution

$(x_1, x_2) = (\frac{36}{11}, \frac{40}{11}) = (3.27, 3.64)$ $z = 30.18$

Drop first constraint, integer solution

$(x_1, x_2) = (3, 4)$ $z = 29.00$

Drop first constraint, LP-relaxed solution

$(x_1, x_2) = (\frac{16}{5}, 4) = (3.2, 4)$ $z = 30.40$

Maximum on Q, LP-relaxed solution

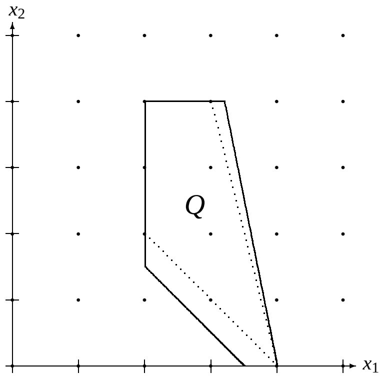
$(x_1, x_2) = (3, 4)$ $z = 29.00$

Maximum on Q, with first constraint added

$(x_1, x_2) = (\frac{28}{9}, \frac{32}{9}) = (3.11, 3.56)$ $z = 28.88$

Geom. interpretation, Lagrangian Relaxation

Viewpoint 1: fixed λ



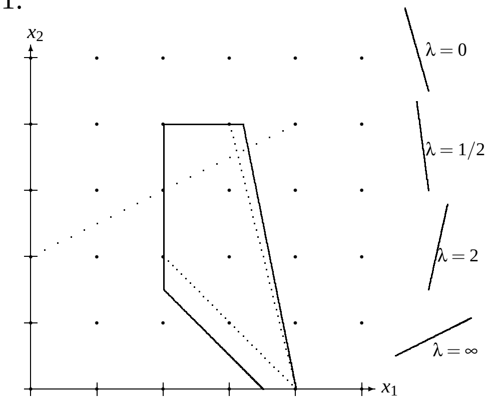
$$\begin{array}{llcccl}
\max & (7+\lambda)x_1+(2-2\lambda)x_2+4\lambda \\
\text{s.t.} & 5x_1 & + & x_2 & \leq & 20 \\
 & -2x_1 & - & 2x_2 & \leq & -7 \\
 & -x_1 & & & \leq & -2 \\
 & & & x_2 & \leq & 4 \\
 & & x_1, x_2 \text{ integer}
\end{array}$$

Redefinition using convex hull of Q

$$\begin{array}{ll}
\max & (7+\lambda)x_1+(2-2\lambda)x_2+4\lambda \\
\text{s.t.} & \left.\begin{array}{rcrcr} 4x_1 & + & x_2 & \leq & 16 \\ -x_1 & - & x_2 & \leq & -4 \\ -x_1 & & & \leq & -2 \\ & & x_2 & \leq & 4 \end{array}\right\} Q
\end{array}$$

Geom. interpretation, Lagrangian Relaxation

Viewpoint 1:



- λ is a modifier of the objective function
- For $0 \leq \lambda \leq \frac{1}{9}$, optimal solution $(3,4)$

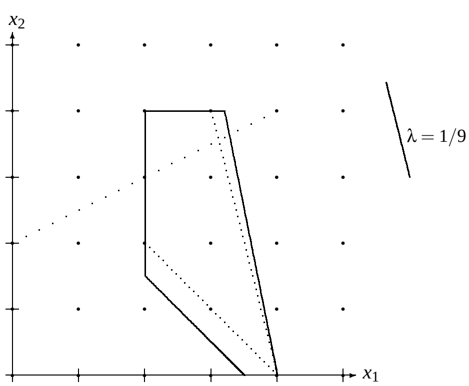
$$z_{LR}(\lambda) = (7+\lambda)3+(2-2\lambda)4+4\lambda = 29-\lambda$$

- For $\lambda \geq \frac{1}{9}$ optimal solution $(4,0)$

$$z_{LR}(\lambda) = (7+\lambda)4+(2-2\lambda)0+4\lambda = 28+8\lambda$$

- Increasing lambda is forcing the optimal solution to satisfy relaxed constraint.

Geom. interpretation, Lagrangian Relaxation



- When $\lambda = \frac{1}{9}$ we get the tightest bound.
- In this case the isoprofit line is parallel to the line through $(3,4)$ and $(4,0)$.
- We may choose an arbitrary point x^* on this line

$$(x_1^*, x_2^*) = (\frac{28}{9}, \frac{32}{9}) = (3.11, 3.56)$$

which satisfies the relaxed constraint

$$-x_1 + 2x_2 \leq 4$$

- In this case

$$z_{LD} = \max\{cx : Dx \leq d, x \in \text{conv}(Q)\}$$

This "proves" theorem 10.3 page 172.

Geom. interpretation, Lagrangian Relaxation

Integer Programming Problem

$$\begin{array}{ll}
\text{maximize} & cx \\
\text{subject to} & Ax \leq b \\
 & Dx \leq d \\
 & x_j \in \mathbb{Z}_+, \quad j = 1,\ldots,n
\end{array}$$

$$\boxed{\max\left\{cx : x \in \text{conv}(Ax \leq b, Dx \leq d, x \in \mathbb{Z}_+)\right\}}$$

Lagrange Relaxation, multipliers $\lambda \geq 0$

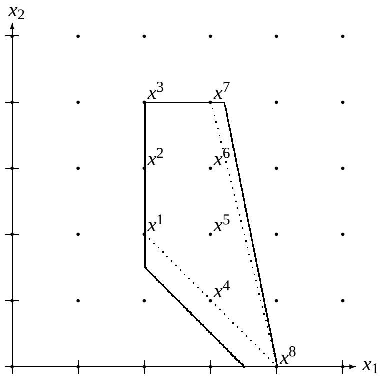
$$\begin{array}{ll}
\text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\
\text{subject to} & Ax \leq b \\
 & x_j \in \mathbb{Z}_+, \quad j = 1,\ldots,n
\end{array}$$

for best multiplier $\lambda \geq 0$

$$\boxed{\max\left\{cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+)\right\}}$$

Geom. interpretation, Lagrangian Relaxation

Viewpoint 2: fixed point x^i



There are 8 integer points in Q:

$$\{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8\} =$$
$$\{(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,0)\}$$

For fixed x^i the objective function

$$z_{LR}(\lambda, x^i) = (7+\lambda)x_1^i + (2-2\lambda)x_2^i + 4\lambda = 7x_1^i + 2x_2^i + \lambda(x_1^i - 2x_2^i + 4)$$

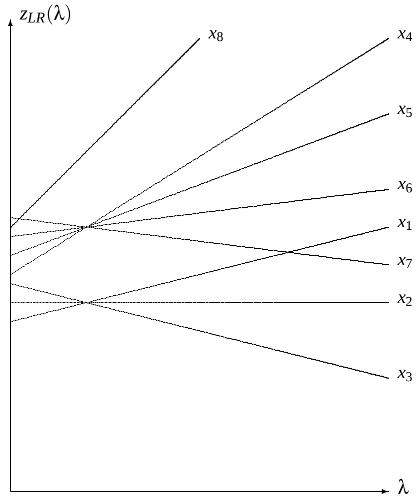
is an affine function.

E.g. for $x^7 = (3,4)$

$$z_{LR}(\lambda, x^7) = 7\cdot 3 + 2\cdot 4 + \lambda(3 - 2\cdot 4 + 4) = 29 - \lambda$$

Geom. interpretation, Lagrangian Relaxation

Viewpoint 2:



Objective

$$z_{LR}(\lambda) = \max_{x^i \in Q} z(\lambda, x^i)$$

Proposition 2 The Lagrangian relaxed problem $z_{LR}(\lambda)$ as function of the multipliers $\lambda \in \mathbb{R}$, $\lambda \geq 0$ is piecewise linear and convex

(see Wolsey, figure page 173)

Lagrangian relaxation and duality

- Lagrangian relaxation is a generalization of duality, where we may "dualize" any subset of constraints.

- Lagrange Relaxation

$$\begin{array}{ll} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \leq b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

  Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

  is an LP-problem

- Optimal multipliers λ may be found by simplex.

- Subgradient is however faster when few iterations.

Lagrangian Relaxation

Integer Programming Problem

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & Dx \leq d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Lagrange Relaxation, multipliers $\lambda \geq 0$

$$\begin{array}{ll} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \leq b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

Assume that the "nice constraints" $Ax \leq b$ define the convex hull, e.g.

- A is totally unimodular, and b is a vector of integers
- There are no constraints left
- The remaining constraints are defined in linear variables

Lagrangian Relaxation

for best multiplier $\lambda \geq 0$ strength of model

$$\boxed{\max\left\{cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+)\right\}}$$

If $\{x : Ax \leq b\} = \{x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+)\}$ strength

$$\boxed{\max\left\{cx : Dx \leq d, Ax \leq b\right\}}$$

Corollary (page 173 in Wolsey)

$$z_{LD} = z_{LP}$$

for any objective function cx.

- We do not obtain better bounds than by linear relaxation.
- We may find $z_{LP} = z_{LD}$ in polynomial time.
- If the remaining problem $Ax \leq b$ has a nice structure (e.g. min-spanning-tree) we may find z_{LD} faster than z_{LP}.

Lagrangian Relaxation

Lagrangian dual when $Ax \leq b$ define convex hull

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & Dx \leq d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Consider LP-relaxation with solution z_{LP}

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & Dx \leq d \\ & x \geq 0 \end{array}$$

where the dual problem is

$$\begin{array}{ll} \text{minimize} & by + dy' \\ \text{subject to} & yA + y'D \geq c \\ & y, y' \geq 0 \end{array}$$

Now consider the Lagrangian relaxed problem $z_{LR}(\lambda)$

$$\begin{array}{ll} \lambda d + \text{maximize} & (c - \lambda D)x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \quad (x \in \mathbb{Z}_+ \text{ for free}) \end{array}$$

where the dual problem is

$$\begin{array}{ll} \lambda d + \text{minimize} & by \\ \text{subject to} & yA \geq c - \lambda D \\ & \lambda, y \geq 0 \end{array}$$

Lagrangian Relaxation

Integer Programming Problem

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & Dx \leq d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

If $Ax \leq b$ define convex hull, solution to Lagrangian dual

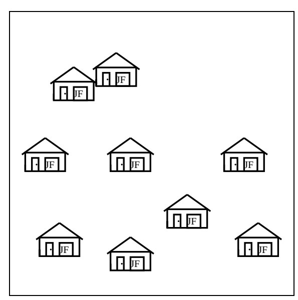
$$\lambda = y'$$

Lagrangian relaxation

- If relax all constraints: ordinary dual problem
- Lagrangian relaxation of a constraint can be seen as "dualization" of a constraint.
- We have found a technique for deriving the best Lagrangian multipliers in some special cases.

Example: A location problem

Dispersion problem: Open p out of n possible facilities so that their overall distance is maximized



- Distance i to j is $d_{ij} \geq 0$.
- $d_{ij} = d_{ji}$ and $d_{jj} = 0$.
- Binary variable x_j is one if facility open

p-dispersion problem

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} \sum_{i=1}^{n} d_{ij} x_i x_j \\ \text{subject to} & \sum_{j=1}^{n} x_j = p \\ & x_j \in \{0, 1\}, \quad j = 1, \ldots, n. \end{array}$$

Better linear formulation

Quadratic model

$$\begin{aligned}\text{maximize} \quad & \sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij}x_i x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = p \\ & x_j \in \{0,1\}\end{aligned}$$

To model $y_{ij} = 1 \Rightarrow (x_i = 1 \text{ and } x_j = 1)$

Introduce $(y_{ij} = 1 \Rightarrow x_j = 1)$ and $(y_{ij} = 1 \Leftrightarrow y_{ji} = 1)$

$$y_{ij} \le x_j, \qquad y_{ij} = y_{ji},$$

Multiply $\sum_{i=1}^{n} x_i = p$ by x_j for each j getting

$$\sum_{i=1}^{n} x_i x_j = \sum_{i=1}^{n} y_{ij} = p x_j \qquad j = 1, \ldots, n$$

Better linear formulation

Linear model

$$\begin{aligned}\text{maximize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}y_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = p \\ & \sum_{i=1}^{n} y_{ij} = p x_j && j = 1, \ldots, n \\ & y_{ij} = y_{ji} && i, j = 1, \ldots, n \\ & y_{ij} \le x_j && i, j = 1, \ldots, n \\ & x_j, y_{ij} \in \{0,1\}\end{aligned}$$

Relaxation: drop $y_{ij} = y_{ji}$

Better linear formulation

Relaxed linear model

$$\begin{aligned}\text{maximize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}y_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = p \\ & \sum_{i=1}^{n} y_{ij} = p x_j && j = 1, \ldots, n \\ & y_{ij} \le x_j && i, j = 1, \ldots, n \\ & x_j, y_{ij} \in \{0,1\}\end{aligned}$$

$$\begin{array}{llllllll}
\max & \boxed{\sum_{i=1}^{n} d_{i1}y_{i1}} & + \boxed{\sum_{i=1}^{n} d_{i2}y_{i2}} & + \boxed{\sum_{i=1}^{n} p_{i3}y_{i3}} & + \ldots + 0x_1 & + 0x_2 & + 0x_3 \; + \ldots & \\
\text{s.t.} & \boxed{\sum_{i=1}^{n} y_{i1}} & & & -px_1 & & & = 0 \\
 & & \boxed{\sum_{i=1}^{n} y_{i2}} & & & -px_2 & & = 0 \\
 & & & \boxed{\sum_{i=1}^{n} y_{i3}} & & & -px_3 & = 0 \\
 & & & & x_1 & +x_2 & +x_3 + \ldots & = p
\end{array}$$

p dispersion problem, deriving the bound

$i \backslash j$	1	2	3	4	5	6	7
1	0	3	7	4	10	5	7
2	3	0	9	5	5	10	6
3	7	9	0	1	3	2	4
4	4	5	1	0	1	9	1
5	10	5	3	1	0	3	2
6	5	10	2	9	3	0	3
7	7	6	4	1	2	3	0

$n = 7,\ p = 3.$

$d'_1 = 24\ d'_2 = 25\ d'_3 = 20\ d'_4 = 18\ d'_5 = 18\ d'_6 = 24\ d'_7 = 17$

Upper bound d'_j on each facility j

$$\begin{aligned}\text{maximize} \quad & d'_j = \sum_{i=1}^{n} d_{ij}y_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{n} y_{ij} = p \\ & y_{ij} \in \{0,1\}, \quad i = 1, \ldots, n.\end{aligned}$$

Upper bound $\bar{z}$

$$\begin{aligned}\text{maximize} \quad & \bar{z} = \sum_{j=1}^{n} d'_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = p \\ & x_j \in \{0,1\}, \quad j = 1, \ldots, n.\end{aligned}$$

p dispersion problem, Lagrangian relaxation

Linear model

$$
\begin{aligned}
\text{maximize}\quad & \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}y_{ij} \\
\text{subject to}\quad & \sum_{j=1}^{n} x_j = p \\
& \sum_{i=1}^{n} y_{ij} = p x_j \qquad j = 1,\ldots,n \\
& y_{ij} = y_{ji} \qquad i \le j \\
& y_{ij} \le x_j \qquad i,j = 1,\ldots,n \\
& x_j, y_{ij} \in \{0,1\}
\end{aligned}
$$

Lagrange relax constraints $y_{ij} = y_{ji}$, using multipliers λ_{ij}.
For symmetry reasons $\lambda_{ij} = -\lambda_{ji}$

p dispersion problem, Lagrangian relaxation

$$
\begin{aligned}
\text{maximize}\quad & \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}y_{ij} - \sum_{i=1}^{n}\sum_{j=i}^{n} \lambda_{ij}(y_{ij} - y_{ji}) \\
& = \sum_{i=1}^{n}\sum_{j=1}^{n} (d_{ij} - \lambda_{ij}) y_{ij} \\
\text{subject to}\quad & \sum_{j=1}^{n} x_j = p \\
& \sum_{i=1}^{n} y_{ij} = p x_j \qquad j = 1,\ldots,n \\
& y_{ij} \le x_j \qquad i,j = 1,\ldots,n \\
& x_j, y_{ij} \in \{0,1\}
\end{aligned}
$$

p dispersion problem, deriving the bound

i \ j	1	2	3	4	5	6	7
1	0	3	7	4	10	5	7
2	3	0	9	5	5	10	6
3	7	9	0	1	3	2	4
4	4	5	1	0	1	9	1
5	10	5	3	1	0	3	2
6	5	10	2	9	3	0	3
7	7	6	4	1	2	3	0

$n = 7$, $p = 3$.

$d'_1 = 24$ $d'_2 = 25$ $d'_3 = 20$ $d'_4 = 18$ $d'_5 = 18$ $d'_6 = 24$ $d'_7 = 17$
$\overline{z} = 73$

i \ j	1	2	3	4	5	6	7
1	0	3	5	3	13	5	11
2	3	0	12	3	3	10	5
3	9	6	0	1	2	2	4
4	5	7	1	0	1	5	1
5	7	7	4	1	0	3	2
6	5	10	2	13	3	0	3
7	3	7	4	1	2	3	0

$n = 7$, $p = 3$.

$d'_1 = 21$ $d'_2 = 24$ $d'_3 = 21$ $d'_4 = 19$ $d'_5 = 19$ $d'_6 = 20$ $d'_7 = 20$
$\overline{z} = 66$

Branch-and-bound tests

GEO *geometrical problems*
d_{ij} Euclidean distance between i and j

WGEO *weighted geometrical problems*
Each facility has a weight. d_{ij} Euclidean distance between i and j times weights

EXP *exponential distribution*
d_{ij} with $i < j$ is randomly drawn from exponential distribution.

AEXP *asymmetric exponential distribution*
as above but $d_{ij} \neq d_{ji}$

RAN *random distances*
d_{ij} randomly distributed in $[1 \ldots 100]$.

DSUB *dense subgraph*
d_{ij} is set to 1 or 0 with 50% probability.

Branch-and-bound results

n	GEO	WGEO	EXP	AEXP	RAN	DSUB
10	13.06	11.72	19.47	41.96	11.51	15.15
20	15.29	10.50	23.91	35.60	17.83	23.51
30	16.10	11.09	25.31	34.48	17.77	22.52
40	15.41	7.03	23.56	29.41	11.23	13.43
50	14.75	10.10	24.20	23.88	26.54	34.68
60	28.85	8.68	32.43	36.84	—	—
70	18.32	9.90	—	—	—	—
80	12.03	9.30	—	—	—	—
90	—	9.48	—	—	—	—
100	—	17.45	—	—	—	—
150	—	8.01	—	—	—	—

Table 1: Relative deviation of upper bound $\bar{z}$ in pct. Deleting constraint. Average of 10 instances.

n	GEO	WGEO	EXP	AEXP	RAN	DSUB
10	7.66	6.76	7.90	7.57	7.68	8.30
20	9.04	4.03	11.11	11.60	11.94	16.37
30	9.15	3.25	14.84	13.40	12.88	16.58
40	8.59	1.47	14.97	13.45	8.64	11.24
50	9.07	3.39	16.17	9.77	20.89	27.86
60	19.69	2.68	22.31	15.69	15.52	—
70	10.80	3.20	—	—	—	—
80	6.93	2.76	—	—	—	—
90	—	2.92	—	—	—	—
100	—	7.09	—	—	—	—
150	—	2.38	—	—	—	—
200	—	2.78	—	—	—	—

Table 2: Relative deviation of upper bound $\bar{z}$ in pct. Lagrange relaxing constraint. Average of 10 instances.

n	GEO	WGEO	EXP	AEXP	RAN	DSUB
10	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00
30	0.01	0.01	0.01	0.01	0.02	0.02
40	0.05	0.01	0.11	0.31	0.57	1.60
50	0.64	0.03	2.79	0.58	12.65	30.47
60	2.85	0.05	87.28	61.52	4552.81	—
70	39.18	0.09	—	—	—	—
80	153.15	0.17	—	—	—	—
90	—	0.33	—	—	—	—
100	—	0.44	—	—	—	—
150	—	3.08	—	—	—	—
200	—	161.21	—	—	—	—

Table 3: Solution times in seconds as average of 10 instances.

n	GEO	WGEO	EXP	AEXP	RAN	DSUB
10	12	7	8	7	9	6
20	140	18	81	227	328	448
30	1654	45	2082	2659	5716	8420
40	12675	26	42851	141162	276980	927312
50	220355	858	1105817	218292	5565562	16162737
60	816524	554	28536918	19629045	3217643	—
70	9727736	1241	—	—	—	—
80	28711239	7282	—	—	—	—
90	—	16652	—	—	—	—
100	—	13646	—	—	—	—
150	—	123478	—	—	—	—
200	—	7302184	—	—	—	—

Table 4: Number of branch-and-bound nodes. Average of 10 instances.