Introduction to Optimization, DIKU 2005/06
Friday, December 16
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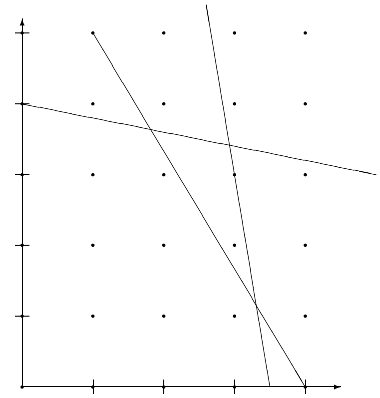
Continue from last time:

- Cutting planes — a method to obtain tighter bounds and faster convergence to integer solutions (Wolsey chap. 8)
- Start on Wolsey Chapter 9

Cuts and facets



Definitions

- cuts: valid inequalities
- facets: inequalities defining convex hull

Cuts and facets are redundant for IP formulation
Tighten formulation for LP relaxation

Overview of cuts

- Chvatal cuts
- Gomory cuts (Modular cuts)
- Chvatal-Gomory cuts
- Disjunctive cuts
- Cover inequalities
- Clique inequalities
- Problem specific cuts

Chvátal Cuts

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{1j} x_j \le b_1 \\ & \vdots \\ & \sum_{j=1}^{n} a_{mj} x_j \le b_m \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{aligned}$$

1 Take a linear combination of the constraints

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} u_i a_{ij}\right) x_j \le \left(\sum_{i=1}^{m} u_i b_i\right)$$

in short

$$\sum_{j=1}^{n} a'_j x_j \le b'$$

2 Divide through by a common factor $d | a'_j, j = 1, \ldots, n$

$$\sum_{j=1}^{n} \frac{a'_j}{d} x_j \le \frac{b'}{d}$$

3 Since all a'_j/d are integers round down b'

$$\sum_{j=1}^{n} \frac{a'_j}{d} x_j \le \lfloor \frac{b'}{d} \rfloor$$

Chvatal-Gomory cuts (p. 119)

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \sum_{j=1}^{n} a_{1j} x_j \leq b_1 \\ & \vdots \\ & \sum_{j=1}^{n} a_{mj} x_j \leq b_m \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

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$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} u_i a_{ij}\right) x_j \leq \left(\sum_{i=1}^{m} u_i b_i\right)$$

in short

$$\sum_{j=1}^{n} a'_j x_j \leq b'$$

2 Since $x \geq 0$ implies $\sum_{j=1}^{n}(a'_j - \lfloor a'_j \rfloor) x_j \geq 0$ we have

$$\sum_{j=1}^{n} \lfloor a'_j \rfloor x_j \leq b'$$

3 Since $x_j \in \mathbb{Z}_+$ implies $\lfloor a'_j \rfloor x_j \in \mathbb{Z}$ we get

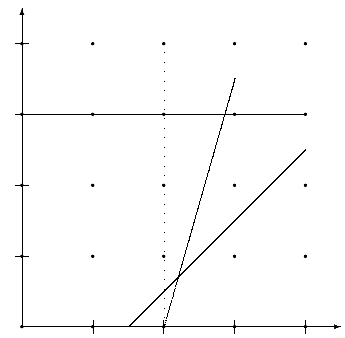
$$\sum_{j=1}^{n} \lfloor a'_j \rfloor x_j \leq \lfloor b' \rfloor$$

Gomory Cuts

- Systematical way of generating valid inequalities
- In each step the current LP-solution will be separated
- Ensures that an integer solution will be reached after a number of steps

Example

$$\begin{array}{lrcrcl} \text{maximize} & 4x_1 & - & x_2 & & \\ \text{subject to} & 7x_1 & - & 2x_2 & \leq & 14 \\ & & & x_2 & \leq & 3 \\ & 2x_1 & - & 2x_2 & \leq & 3 \\ & x_1, & & x_2 & \geq & 0, \text{ integer} \end{array}$$



Gomory Cuts - example

Adding slack variables $x_3, x_4, x_5 \geq 0$, and solving LP-problem (Taha simplex table)

basis	z	x_1	x_2	x_3	x_4	x_5	solution
z	1			$\frac{4}{7}$	$\frac{1}{7}$		$\frac{59}{7}$
x_1		1		$\frac{1}{7}$	$\frac{2}{7}$		$\frac{20}{7}$
x_2			1		1		3
x_5				$-\frac{2}{7}$	$\frac{10}{7}$	1	$\frac{23}{7}$

The simplex table as equations

$$A_B x_B + A_N x_N = b$$

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$\begin{array}{lcccccccl} \text{max} & \frac{59}{7} & & & - \frac{4}{7}x_3 & - \frac{1}{7}x_4 & & & \\ \text{s.t.} & & x_1 & & + \frac{1}{7}x_3 & + \frac{2}{7}x_4 & & = & \frac{20}{7} \\ & & & x_2 & & + x_4 & & = & 3 \\ & & & & - \frac{2}{7}x_3 & + \frac{10}{7}x_4 & + x_5 & = & \frac{23}{7} \\ & & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0, \text{ integer} \end{array}$$

The optimal LP-solution is

$$(x_1, x_2, x_3, x_4, x_5) = \left(\frac{20}{7}, 3, 0, 0, \frac{23}{7}\right)$$

which is fractional.

From first equation in Simplex table we get

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7}$$

and hence also $x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 \leq \frac{20}{7}$, so

$$x_1 + \left\lfloor \frac{1}{7} \right\rfloor x_3 + \left\lfloor \frac{2}{7} \right\rfloor x_4 \leq \left\lfloor \frac{20}{7} \right\rfloor$$

inserting $x_1 = -\frac{1}{7}x_3 - \frac{2}{7}x_4 + \frac{20}{7}$ we get

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

or substituting the slack variables x_3 and x_4 we get

$$\frac{1}{7}(14 - 7x_1 + 2x_2) + \frac{2}{7}(3 - x_2) \geq \frac{6}{7}$$

which can be reduced to $x_1 \leq 2$.

Gomory Cuts

Gomory (1963) presented a general technique for solving IP problems

1 Solve the LP-relaxation

2 Choose one of the basis integer variables taking a fractional value

$$x_i + \sum_{j \in N} a_j x_j = a_0 \quad (1)$$

3 Use the corresponding equation to separate the inequality

$$\sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j \geq (a_0 - \lfloor a_0 \rfloor) \quad (2)$$

4 Incorporate the new constraint and repeat.

Proposition 1 Inequality (2) is a valid inequality which separates the current LP solution from the feasible set.

Proof

$$x_i + \sum_{j \in N} a_j x_j = a_0 \quad (1)$$

$$\sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j \geq (a_0 - \lfloor a_0 \rfloor) \quad (2)$$

- [The inequality (2) is valid]

  Since (1) is valid we also have

  $$x_i + \sum_{j \in N} a_j x_j \leq a_0$$

  Derive a C-G cut

  $$x_i + \sum_{j \in N} \lfloor a_j \rfloor x_j \leq \lfloor a_0 \rfloor$$

  substitute $x_i = -\sum_{j \in N} a_j x_j + a_0$ getting

  $$-\sum_{j \in N} a_j x_j + a_0 + \sum_{j \in N} \lfloor a_j \rfloor x_j \leq \lfloor a_0 \rfloor$$

  or

  $$a_0 - \lfloor a_0 \rfloor \leq \sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j$$

- [Inequality (2) separates current solution]

  Current solution was $x_i = a_0$ and $x_j = 0, j \in N$. Inserted in equation (2)

  $$\sum_{j \in N} (a_j - \lfloor a_j \rfloor) 0 \geq (a_0 - \lfloor a_0 \rfloor) > 0$$

Lexicographic order

Given a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$

$v \underset{L}{>} 0$ v is `lex-positive` if first $v_i \neq 0$ is positive

$v \underset{L}{=} 0$ v is `lex-zero` if $v_i = 0, i = 1, \ldots, n$

$v \underset{L}{<} 0$ v is `lex-negative` if first $v_i \neq 0$ is negative

Given two vectors $v, w \in \mathbb{R}^n$

$v \underset{L}{<} w$ v is `lex-less-than` w if $v - w \underset{L}{<} 0$

$v \underset{L}{>} w$ v is `lex-greater-than` w if $v - w \underset{L}{>} 0$

Define `lex-min`, `lex-max` obvious way

Example

$$(0,0,1,0) \underset{L}{>} (0,0,0,2)$$

$$(0,3,1,2) \underset{L}{<} (1,2,4,8)$$

Lexicographic Anti-cycling Rule for Simplex

The primal simplex algorithm terminates after a finite number of pivots if

- Entering variable: choose any column s with $a_{0s} < 0$
- Leaving variable: choose row by

  $$\underset{i\,:\,a_{is} > 0}{\texttt{lex-min}} \frac{a_i}{a_{is}}$$

  where $a_i = (\text{solution}_i, a_{i1}, a_{i2}, \ldots, a_{in})$

Example (maximization)

basis	z	x_1	x_2	x_3	x_4	x_5	x_6	solution
z	1				-3	1	2	15
x_1		1			4	1	3	1
x_2			1		1	10	1	2
x_3				1	12	1	2	3

Entering variable $s = 4$ since $a_{0s} = -3$. Leaving variable

$$\frac{a_1}{a_{1s}} = (\tfrac{1}{4}, 1, \tfrac{1}{4}, \tfrac{3}{4}) \quad \frac{a_2}{a_{2s}} = (2, 1, 10, 1) \quad \frac{a_3}{a_{3s}} = (\tfrac{1}{4}, 1, \tfrac{1}{12}, \tfrac{1}{6})$$

Choose row 3.

Gomory Cuts

For pure IP-models we have

Proposition

If we always

- derive the Gomory cut from the first simplex row in which the basis variable is fractional
- use the lexicographic version of the simplex algorithm

then Gomory's fractional algorithm will find an integer optimal solution in a finite number of steps

However

There is no polynomial bound on the number of steps

Modular Inequalities

Valid inequalities for

$$S = \left\{ x \in \mathbb{Z}_+^n : \sum_{j \in N} a_j x_j = a_0 \right\}$$

Extend S to all points which satisfy the inequality plus kd, where $k > 0$, integer, $d \geq 1$, integer.

$$S_d = \left\{ x \in \mathbb{Z}_+^n : \sum_{j \in N} a_j x_j = a_0 + kd, \text{ some integer } k \right\}$$

Let b_j be the remainder when a_j is divided by d. Thus

$$a_j = b_j + \alpha_j d$$

where $0 \leq b_j < d$ and α_j integer. Then

$$S_d = \left\{ x \in \mathbb{Z}_+^n : \sum_{j \in N} b_j x_j = b_0 + kd, \text{ some integer } k \right\}$$

The integer k must satisfy

$$\begin{aligned} k &= \tfrac{\sum b_j x_j}{d} - \tfrac{b_0}{d} & \\ &\geq 0 - \tfrac{b_0}{d} & \text{since } \textstyle\sum_{j \in N} b_j x_j \geq 0 \\ &> -1 & \text{since } b_0/d < 1 \\ &\geq 0 & \text{since } k \text{ integer} \end{aligned}$$

Thus we have the valid inequality

$$\sum_{j \in N} b_j x_j \geq b_0$$

Since $S \subset S_d$, inequality is valid for S.

Disjunctive Inequalities (section 8.8)

Proposition Assume that

$$\sum_{j \in N} \pi_j x_j \leq \pi_0$$

is a valid inequality for X_1 and

$$\sum_{j \in N} \pi'_j x_j \leq \pi'_0$$

is a valid inequality for X_2. Then

$$\sum_{j \in N} \min(\pi_j, \pi'_j) x_j \leq \max(\pi_0, \pi'_0)$$

is a valid inequality for $X_1 \cup X_2$.

Proof If we have the valid inequality

$$\sum_{j \in N} \pi_j x_j \leq \pi_0$$

then also

$$\sum_{j \in N} b_j x_j \leq b_0$$

is a valid inequality if $b_j \leq \pi_j$ and $b_0 \geq \pi_0$. $\square$

Example

Valid inequality for X_1

$$2x_1 + 3x_2 + 7x_3 \leq 15$$

Valid inequality for X_2

$$4x_1 + 2x_2 + 9x_3 \leq 11$$

Then valid inequality for $X_1 \cup X_2$ is

$$2x_1 + 2x_2 + 7x_3 \leq 15$$

Wolsey chapter 9

Deeper understanding of cuts, facets

- We would like to use the “best” formulation
- Dominance, redundance, facets
- Facets are intuitively easy to understand
- How to prove that a valid inequality is facet defining?

Notice

$$\pi x \le \pi_0$$

and

$$\lambda\pi x \le \lambda\pi_0$$

are identical for any $\lambda > 0$

Dominance

$$\begin{array}{llcllcl} \text{maximize} & \ldots \\ \text{subject to} & 1x_1 & + & 3x_2 & \le & 4 \\ & 2x_1 & + & 4x_2 & \le & 9 \\ & x_1, x_2 \ge 0 \end{array}$$

Multiplying the second inequality with $u = \frac{1}{2}$

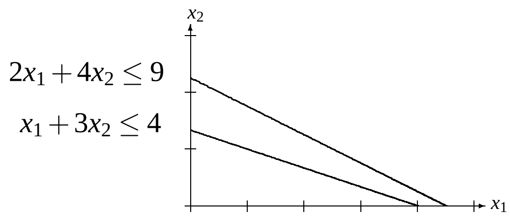
$$1x_1 + 2x_2 \le \frac{9}{2}$$

First inequality dominates the second.

Dominance:

$$\pi x \le \pi_0 \qquad \mu x \le \mu_0$$

$\pi x \le \pi_0$ *dominates* $\mu x \le \mu_0$ if there exists $u > 0$ such that $\pi \ge u\mu$ and $\pi_0 \le u\mu_0$.



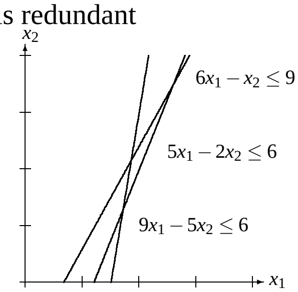
Redundance

$$\begin{array}{llcllcl} \text{maximize} & \ldots \\ \text{subject to} & 6x_1 & - & x_2 & \le & 9 \\ & 9x_1 & - & 5x_2 & \le & 6 \\ & 5x_1 & - & 2x_2 & \le & 6 \\ & x_1, x_2 \ge 0 \end{array}$$

Multiplying the first two constraints with $u = (\frac{1}{3}, \frac{1}{3})$

$$5x_1 - 2x_2 \le 5$$

Last inequality is redundant



Redundance:

$$\pi^i x \le \pi^i_0, \ \ i = 1, \ldots, k$$

$$\mu x \le \mu_0$$

Inequality $\mu x \le \mu_0$ is *redundant* if there exists a vector $(u_1, \ldots, u_k) \ge 0$ such that

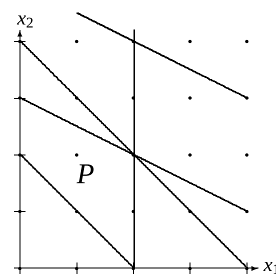
$$(\sum_{i=1}^{k} u_i \pi^i) x_i \le \sum_{i=1}^{k} u_i \pi^i_0$$

dominates $\mu x \le \mu_0$

Polyhedra, Facets

Polyhedra $P \subset \mathbb{R}^2$

$$\begin{array}{llcllcl} \text{subject to} & x_1 & & & \le & 2 \\ & x_1 & + & x_2 & \le & 4 \\ & x_1 & + & 2x_2 & \le & 10 \\ & x_1 & + & 2x_2 & \le & 6 \\ & x_1 & + & x_2 & \ge & 2 \\ & x_1, x_2 \ge 0 \end{array}$$



- $P \subset \mathbb{R}^2$ and “both directions are present”
- P is full-dimensional.
- The points $(2,0)$, $(1,1)$ and $(2,2)$ are affinely independent points.
- The vectors $(2,0,1)$, $(1,1,1)$ and $(2,2,1)$ are linearly independent.
- The dimension of P is one less than the number of affinely independent points.

Polyhedra, Facets

Affinely independent

The points $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ are affinely independent if the $k-1$ directions $(x^2 - x^1), \ldots, (x^k - x^1)$ are linearly independent.

Dimension

The dimension of P, denoted $\dim(P)$ is one less than the maximum number of affinely independent points in P.

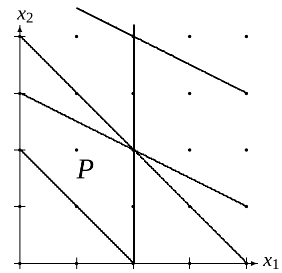
A line is 1-dim
A plane is 2-dim
A box is 3-dim

Full-dimensional

The polyhedra $P \subseteq \mathbb{R}^n$ is full-dimensional if and only if $\dim(P) = n$.

Polyhedra, Facets

$$
\begin{array}{lrcrcr}
\text{subject to} & x_1 & & & \leq & 2 \\
& x_1 & + & x_2 & \leq & 4 \\
& x_1 & + & 2x_2 & \leq & 10 \\
& x_1 & + & 2x_2 & \leq & 6 \\
& x_1 & + & x_2 & \geq & 2 \\
& x_1, x_2 \geq 0 & & & &
\end{array}
$$



- $x_1 \leq 2$ defines a facet of P, as $(2,0)$ and $(2,2)$ are two affinely independent points in P.
- $x_1 + 2x_2 \leq 6$ defines a facet.
- $x_1 + x_2 \geq 2$ defines a facet.
- $x_1 \geq 0$ defines a facet.
- $x_1 + x_2 \leq 4$ is a face with one point $(2,2) \in P$
- $x_1 + x_2 \leq 4$ is redundant: $u = (\frac{1}{2}, 0, 0, \frac{1}{2}, 0)$
- $x_2 \geq 0$ is redundant: $u = (1, 0, 0, 0, -1)$

Face, Facets

If $\pi x \leq \pi_0$ is a valid inequality of P then

$$F = \{x \in P : \pi x = \pi_0\}$$

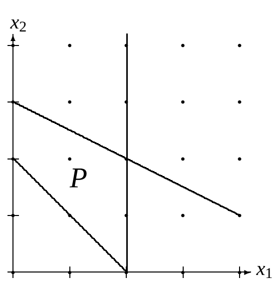
defines a face of P.

F is a facet of P iff

- F is a face of P
- $\dim(F) = \dim(P) - 1$

Minimal description of previous example

$$
\begin{array}{lrcrcr}
\text{subject to} & x_1 & & & \leq & 2 \\
& x_1 & + & 2x_2 & \leq & 6 \\
& x_1 & + & x_2 & \geq & 2 \\
& x_1 & & & \geq & 0
\end{array}
$$

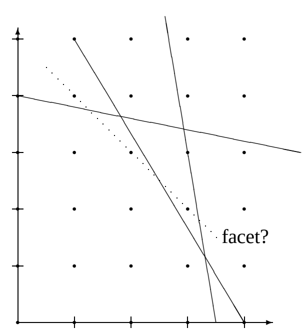


IP-problems

Two-dimensional problem

$$P = \{x : Ax \leq b\} \cap \mathbb{Z}^2$$

is dotted line a facet?



- Dimension of P is 2
- The facet defining inequality must be valid
- A facet should have dimension 1
- There should be 2 affine intendant points on a facet