

Program of the day:

- Convex Hull, Totally Unimodular (Wolsey sec. 3.1-3.2)
- Cutting planes — a method to obtain tighter bounds and faster convergence to integer solutions (Wolsey chap. 8)
- Application: branch-and-cut algorithms

Solving IP models

Some IP naturally lead to integer solutions

- Totally unimodular (TU) matrices

Otherwise, reformulation

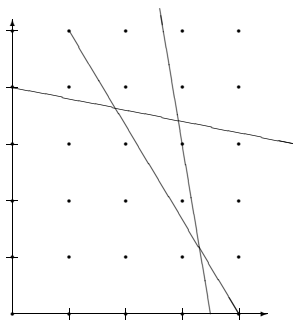
- Reformulation of constraints to TU
- Tightening M, m
- Fixation of variables

Finally, enumerative methods

- Branch-and-bound methods
- Cutting plane methods

Convex hull

Smallest convex polyhedral which contains all integer points



- feasible solutions $\{x \in \mathbb{N}^n : Ax \leq b\}$
- linear relaxation $\{x \in \mathbb{R}^n : Ax \leq b\}$
- convex hull $\text{conv}\{x \in \mathbb{R}^n : Ax \leq b\}$

If constraints of an IP-model define the convex hull, then we can solve the problem efficiently.

Totally Unimodularity

Definition 1 An $m \times n$ integral matrix A is called *totally unimodular* (TU) if the determinant of each square submatrix of A is equal to 0, 1 or -1.

Obviously a_{ij} must be 0, 1, -1

Recognising whether A is TU demands an exponential number of steps

Proposition 1 If A is TU then A_B is also TU

Proof: If A is TU then the determinant of each square submatrix of A is equal to 0, 1 or -1. This also holds when restricted to columns in A_B .

Proposition 2 If A is TU then A^{-1} is also TU

Proof: From Cramer's rule $A_{ij}^{-1} = C_{ji} / \det(A)$ where C_{ji} is the adjoint matrix

$$C_{ji} = (-1)^{i+j} \det(A_{\text{row } i, \text{ column } j \text{ removed}})$$

Proposition 3 If A is TU and b is integral then any basis solution x_B is integral

Proof:

$$x_B = A_B^{-1}b$$

Good and bad formulations (Williams)

- i) The straightforward formulation results in an IP model where the feasible region is already the convex hull of integer points.
- ii) The problem can fairly easily be reformulated to give a feasible region corresponding to the convex hull of integer points.
- iii) By reformulation it is possible to reduce the feasible region of the LP problem to nearer that of the convex hull of integer points.

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Cutting Planes – introduction

- branch-and-bound: divide and conquer.
- cutting plane: add inequalities which separate fractional solution from solution space.

Development

- 50's cutting plane (Gomory: simplex, no \mathcal{NP} -hardness)
- 70's tighten formulation in preprocessing
- 80-90's branch-and-cut (Padberg, Rinaldi)

Preprocessing → part of solution process

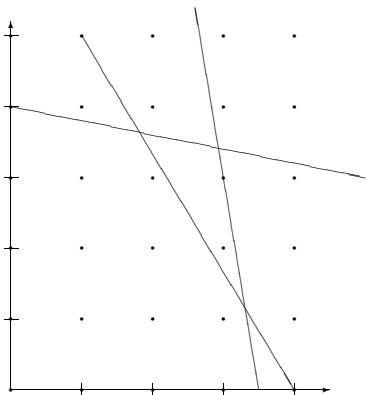
Definitions

- cuts: valid inequalities
- facets: inequalities defining convex hull

Cuts and facets are redundant for IP formulation
Tighten formulation for LP relaxation

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Cuts and facets



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Example

Preprocessing, integer variables

$$\begin{array}{ll} \text{maximize} & \dots \\ \text{subject to} & 7x_1 + 3x_2 - 4x_3 - 2x_4 \leq 1 \\ & -2x_1 + 7x_2 + 3x_3 + 4x_4 \leq 6 \\ & \quad - 2x_2 - 3x_3 - 6x_4 \leq -5 \\ & 3x_1 \quad \quad - 2x_3 \geq -1 \\ & x \in \mathbb{B}^4 \end{array}$$

Generating logical inequalities

From constraint 1 we see that

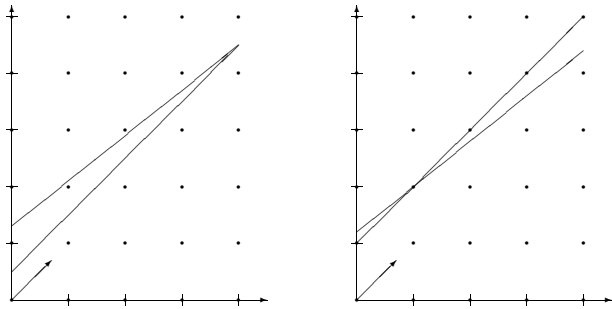
- if $x_1 = 1$ and $x_2 = 1$ then infeasible, thus

$$x_1 + x_2 \leq 1$$

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Example

$$\begin{aligned} &\text{maximize } x_1 + x_2 \\ &\text{subject to } -2x_1 + 2x_2 \geq 1 \\ &\quad -8x_1 + 10x_2 \leq 13 \\ &\quad x_1, x_2 \geq 0, \text{ integer} \end{aligned}$$



Tightening formulation

$$\begin{aligned} -2x_1 + 2x_2 &\geq 1 & -8x_1 + 10x_2 &\leq 13 \\ -x_1 + x_2 &\geq 1/2 & -4x_1 + 5x_2 &\leq 13/2 \\ -x_1 + x_2 &\geq 1 & -4x_1 + 5x_2 &\leq 6 \end{aligned}$$

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Motivation

Integer programming problem (IP)

$$\max\{cx : x \in X\}$$

where $X = \{x : Ax \leq b, x \in \mathbb{Z}_+^n\}$. Reformulate to

$$\max\{cx : x \in \text{conv}(X)\}$$

For any c , an optimal solution to LP is also optimal to IP

Valid inequalities (def. 8.1)

Consider the problem:

$$\begin{aligned} &\text{maximize } f(x) \\ &\text{subject to } x \in X \end{aligned}$$

An inequality

$$\pi x \leq \pi_0$$

is a *valid inequality* for $X \subseteq \mathbb{R}^n$ if

$$\pi x \leq \pi_0 \text{ for all } x \in X$$

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Characterization of valid inequalities (sec. 8.3.2)

Consider the problem:

$$\begin{aligned} &\text{maximize } f(x) \\ &\text{subject to } x \in X \end{aligned}$$

where

$$X = \{y \in \mathbb{Z} : y \leq b\}$$

then the inequality

$$y \leq \lfloor b \rfloor$$

is valid for X

- Simple observation
- Complete characterization

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Overview of cuts

- Chvatal cuts
- Gomory cuts (Modular cuts)
- Chvatal-Gomory cuts
- Disjunctive cuts
- Cover inequalities
- Clique inequalities
- Problem specific cuts

Notice

- Cuts and facets are independent of objective function
- A tight formulation can be used for any objective

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Example of Facets

The problem

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 7x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + 4x_2 + x_3 \geq 10 \\ & 4x_1 + 2x_2 + 2x_3 \geq 13 \\ & x_1 + x_2 - x_3 \geq 0 \\ & x_1, x_2, x_3 \geq 0, \text{ integer} \end{aligned}$$

has the facets

$$\begin{aligned} x_1 + 4x_2 + x_3 &\geq 10 \\ 2x_1 + x_2 + x_3 &\geq 7 \\ x_1 + x_2 - x_3 &\geq 0 \\ x_1 + 3x_2 + x_3 &\geq 9 \\ 2x_1 + 4x_2 + x_3 &\geq 13 \\ x_1 + x_2 + x_3 &\geq 5 \\ x_1 + 2x_2 &\geq 5 \\ 2x_1 + x_2 &\geq 4 \\ x_1 &\geq 0, \text{ integer} \\ x_2 &\geq 0, \text{ integer} \\ x_3 &\geq 0, \text{ integer} \end{aligned}$$

Using the new formulation we obtain an integer optimal solution by solving the LP-relaxed problem. (For any objective function).

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Chvátal Cuts

Valid inequalities for a pure IP-model (minimization)

- 1 Add constraints, using suitable multipliers
- 2 Divide through by a common coefficient factor
- 3 Round up right-hand-side to the next integer

Example

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 7x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + 4x_2 + x_3 \geq 10 \quad (1) \\ & 4x_1 + 2x_2 + 2x_3 \geq 13 \quad (2) \\ & x_1 + x_2 - x_3 \geq 0 \quad (3) \\ & x_1 \geq 0 \quad (4) \\ & x_2 \geq 0 \quad (5) \\ & x_3 \geq 0 \quad (6) \\ & x_1, x_2, x_3 \text{ integer} \end{aligned}$$

1 times (2) is

$$4x_1 + 2x_2 + 2x_3 \geq 13$$

divide by two

$$2x_1 + x_2 + x_3 \geq 6\frac{1}{2}$$

left hand side is integral, thus round up right-hand

$$2x_1 + x_2 + x_3 \geq 7$$

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Example (continued)

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 7x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + 4x_2 + x_3 \geq 10 \quad (1) \\ & 4x_1 + 2x_2 + 2x_3 \geq 13 \quad (2) \\ & x_1 + x_2 - x_3 \geq 0 \quad (3) \\ & x_1 \geq 0 \quad (4) \\ & x_2 \geq 0 \quad (5) \\ & x_3 \geq 0 \quad (6) \\ & x_1, x_2, x_3 \text{ integer} \end{aligned}$$

Facets

$$\begin{aligned} x_1 + 4x_2 + x_3 &\geq 10 \quad (a) \\ 2x_1 + x_2 + x_3 &\geq 7 \quad (b) \\ x_1 + x_2 - x_3 &\geq 0 \quad (c) \\ x_1 + 3x_2 + x_3 &\geq 9 \quad (d) \\ 2x_1 + 4x_2 + x_3 &\geq 13 \quad (e) \\ x_1 + x_2 + x_3 &\geq 5 \quad (f) \\ x_1 + 2x_2 &\geq 5 \quad (g) \\ 2x_1 + x_2 &\geq 4 \quad (h) \\ x_1, x_2, x_3 &\geq 0, \text{ integer} \end{aligned}$$

Obtained as

- (d) : $5 \times (1), 1 \times (b), 1 \times (6)$, divide 7
- (f) : $1 \times (1), 3 \times (b), 3 \times (6)$, divide 7
- (g) : $4 \times (1), 1 \times (b), 5 \times (3)$, divide 11
- (h) : $1 \times (b), 1 \times (c), 1 \times (4)$, divide 2
- (e) : $3 \times (d), 1 \times (b), 3 \times (g)$, divide 4

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Chvatal-Gomory cuts (p. 119)

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^n c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^n a_{1j} x_j \leq b_1 \\ & \vdots \\ & \sum_{j=1}^n a_{mj} x_j \leq b_m \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

1 Take a linear combination of the constraints

$$\sum_{j=1}^n \left(\sum_{i=1}^m u_i a_{ij} \right) x_j \leq \left(\sum_{i=1}^m u_i b_i \right)$$

in short

$$\sum_{j=1}^n a'_j x_j \leq b'$$

2 Since $x \geq 0$ implies $\sum_{j=1}^n (a'_j - \lfloor a'_j \rfloor) x_j \geq 0$ we have

$$\sum_{j=1}^n \lfloor a'_j \rfloor x_j \leq b'$$

3 Since $x_j \in \mathbb{Z}_+$ implies $\lfloor a'_j \rfloor x_j \in \mathbb{Z}$ we get

$$\sum_{j=1}^n \lfloor a'_j \rfloor x_j \leq \lfloor b' \rfloor$$

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Chvatal-Gomory (Theorem 8.4)

$$X = \{x : Ax \leq b, x \in \mathbb{Z}_+^n\}$$

Every valid inequality for X can be obtained by applying the Chvatal-Gomory procedure a finite number of times.

Notice

- No stronger inequalities than Chvatal-Gomory exists.
- Even the facet constraints can be generated as Chvatal-Gomory cuts.
- No constructive (polynomial) algorithm for how the linear combination of constraints should be chosen.
- In practice, the derivation of Chvatal-Gomory cuts must rely on specific features of a given application.

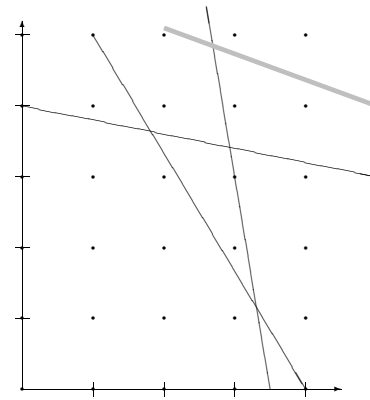
Gomory cuts is a systematical way of deriving cutting planes.

Only 0-1 case

All bounded integer variables can be expressed as sum of binary variables.

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The set $X = P \cap \mathbb{Z}^n$



$$P = \{x \in \mathbb{R}^n : Ax \leq b, 0 \leq x \leq 1\} \neq \emptyset$$

$$X = P \cap \mathbb{Z}^n$$

Nemhauser and Wolsey, Proposition 1.1 page 208:

If inequality $\pi x \leq \pi_0$ is valid for P then it can be obtained as a C-G cut

(*)

- LP-redundant constraints are C-G inequalities
- Theorem 8.4 deals with IP-constraints

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Proof (0-1 case)

Assume that

$$\pi x \leq \pi_0 \text{ where } \pi, \pi_0 \text{ integers}$$

is a valid inequality for X . We will show that this inequality can be obtained by using the C-G procedure a finite number of times.

- Step 1: Find a large number $t \in \mathbb{Z}_+$ such that

$$\pi x \leq \pi_0 + t$$

is a valid C-G inequality

- Step 2: Prove that if

$$\pi x \leq \pi_0 + \tau + 1$$

for $\tau \in \mathbb{Z}_+$ is a C-G inequality for X then also

$$\pi x \leq \pi_0 + \tau$$

is a C-G inequality for X .

- Step 3: Use step 2 for $\tau = t - 1, \dots, 0$ each time getting a new C-G inequality

(Proof by induction)

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Step 1

The inequality

$$\pi x \leq \pi_0 + t$$

is valid for P for some $t \in \mathbb{Z}_+$.

Proof

We have the inequality

$$x \leq 1$$

derive C-G inequality using multipliers $u = \pi$

$$\pi x \leq \pi 1$$

choosing $t = \pi 1 - \pi_0$ (π, π_0 is integer) we get the form

$$\pi x \leq \pi 1 = \pi_0 + t$$

for some $t \in \mathbb{Z}_+$

Note that $t < \infty$ as $P \subseteq [0, 1]^n$ is bounded so $\max\{\pi x \mid x \in P\} < \infty$.

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Step 2

Difficult part

- a) Prove that if $\pi x \leq \pi_0 + \tau + 1$ with $\tau \in \mathbb{Z}_+$ is a C-G cut then

$$\pi x \leq \pi_0 + \tau + \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)$$

is a C-G inequality for X for every partition (N^0, N^1) of $N = \{1, \dots, n\}$.

- b) Use partitionings $(T^0 \cup \{n\}, T^1)$ and $(T^0, T^1 \cup \{n\})$ to obtain a new inequality for (T^0, T^1) .
- c) Derive all valid inequalities for partitionings of $N' = \{1, \dots, n-1\}$
- d) Repeating this procedure n times implies that we eliminate the sums on the right side and thus

$$\pi x \leq \pi_0 + \tau$$

is a C-G cut

Time complexity

- part (c) takes $O(2^n)$,
part (d) is performed n times,
in total $O(n2^n)$
- we run Step 2 $O(t)$ times, thus in total $O(tn2^n)$.

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Step 2, a)

An inequality is valid for P if it is valid for all extreme points $\{x^1, \dots, x^m\}$ of P

Assume that $\pi x \leq \pi_0 + \tau + 1$ with $\tau \in \mathbb{Z}_+$ is a valid cut. Let (N^0, N^1) be any partitioning of $N = \{1, \dots, n\}$. Consider an extreme point x^k of P

- x^k integer: then $\pi x^k \leq \pi_0$ (since $\pi x \leq \pi_0$ valid for X)
- x^k fractional: exists $\varepsilon > 0$ such that

$$\varepsilon^k \leq \sum_{j \in N^0} x_j^k + \sum_{j \in N^1} (1 - x_j^k)$$

Choose $\alpha = \min_{x^k \text{ vertex in } P} \varepsilon^k$

Using $M \geq (\tau + 1)/\alpha$, we have

$$\tau + 1 \leq M\alpha \leq M \left(\sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j) \right)$$

adding π_0 at both sides we get valid inequality for P

$$\pi x \leq \pi_0 + \tau + 1 \leq \pi_0 + M \left(\sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j) \right)$$

Due to (*) the inequality is a C-G cut.

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Step 2, a)

We have just shown that the following is a C-G inequality

$$\pi x \leq \pi_0 + M \left(\sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j) \right)$$

By assumption we had the C-G inequality

$$\pi x \leq \pi_0 + \tau + 1$$

use weights $1/M$ and $(M-1)/M$ for the two inequalities getting C-G inequality

$$\pi x \leq \pi_0 + \tau + \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)$$

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Step 2, b)

Use partitions $(T^0 \cup \{n\}, T^1)$ and $(T^0, T^1 \cup \{n\})$

$$\pi x \leq \pi_0 + \tau + \sum_{j \in T^0 \cup \{n\}} x_j + \sum_{j \in T^1} (1 - x_j)$$

and

$$\pi x \leq \pi_0 + \tau + \sum_{j \in T^0} x_j + \sum_{j \in T^1 \cup \{n\}} (1 - x_j)$$

using multipliers $1/2$ and $1/2$ we get C-G inequality

$$\pi x \leq \pi_0 + \tau + \sum_{j \in T^0} x_j + \sum_{j \in T^1} (1 - x_j)$$

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Branch-and-cut algorithms

Combines best properties from Branch-and-bound and cutting plane.

- Basically a branch-and-bound algorithm
- at each node solve LP-relaxation to find bound
- generate valid inequalities which separate the LP-solution, and which are *valid for the whole problem*
- maintain pool of valid inequalities
- branch when cuts become weak
- convergence ensured by branch-and-bound

Improvements

- Heuristic for generating cut
- Problem specific cuts
- Heuristic for removing cuts

If separation problem is “easy” the cut is not tight