Introduction to Optimization:

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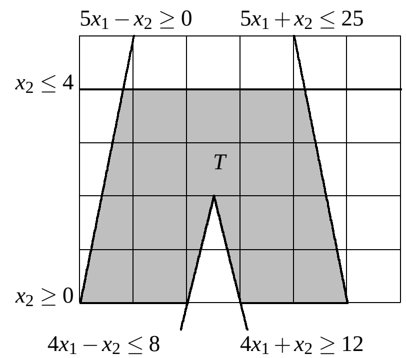
Your assignment

20 different questions Q1-Q20 are posed on the subsequent pages. Q1-Q8 and Q11-Q18 are *multiple choice questions*. For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation to write, for example, “7.b” as your answer to question Q7. Q9-Q10 and Q19-Q20 are ordinary *text questions*. Each correct answer to a multiple choice question gives 4 points, to a text question gives 9 points. The maximum score is thus 100 points.

Note: only the last 10 questions are available

Chippendales

In order to meet the critics of several female students saying that beer does not appeal to women, Tuberg starts to support shows with the Chippendales. The trunks used by Chippendales are designed according to a secret cut T depicted at the following figure:



Q11: To model T as a MIP model the following constraints are obviously necessary

$$x_2 \leq 4$$
$$x_2 \geq 0$$
$$5x_1 - x_2 \geq 0$$
$$5x_1 + x_2 \leq 25$$

Which constraints should be added to the model to get a complete integer linear model of T?

A) $4x_1 - x_2 + 12\delta \leq 20$, $4x_1 + x_2 - 12\delta \geq 0$, $\delta \in \{0,1\}$

B) $4x_1 - x_2 + 12\delta_A \leq 20$, $4x_1 + x_2 - 12\delta_B \geq 0$, $\delta_A + \delta_B \geq 1$, $\delta_A, \delta_B \in \{0,1\}$

C) $4x_1 - x_2 \leq 8$, $4x_1 + x_2 \geq 12$

D) $x_2 + 4|x_1 - \frac{5}{2}| \geq 2$

E) $4x_1 - x_2 + 12\delta_A \leq 20$, $4x_1 + x_2 - 12\delta_B \geq 0$, $\delta_A + \delta_B \leq 1$, $\delta_A, \delta_B \in \{0,1\}$

F) $\delta_A(4x_1 - x_2 \leq 8)$, $\delta_B(4x_1 + x_2 \geq 12)$, $\delta_A + \delta_B \geq 1$, $\delta_A, \delta_B \in \{0,1\}$

Q12: In order to improve sales, the management of Tuberg decides to change the design of the trunks such that it becomes $T' = \text{conv}\{x : x \in T, x \in \mathbb{N}^2\}$. The members of Chippendales are asked to write up an LP-model describing T'. Despite their talents the members end up having written six different models. Which of the formulations is a correct LP-model describing T'?

A)
$$\begin{array}{l} x_2 \leq 4 \\ x_2 \geq 0 \\ 4x_1 - x_2 \geq 0 \\ 4x_1 + x_2 \leq 20 \\ 4x_1 - x_2 + 12\delta_A \leq 20 \\ 4x_1 + x_2 - 12\delta_B \geq 0 \\ \delta_A + \delta_B \leq 1 \\ \delta_A, \delta_B \in \{0,1\} \end{array}$$

B)
$$\begin{array}{l} x_2 \leq 4 \\ x_2 \geq 0 \\ 4x_1 - x_2 \geq 0 \\ 4x_1 + x_2 \leq 20 \\ x_1 + 3\delta_A \leq 5 \\ x_1 - 3\delta_B \geq 0 \\ \delta_A + \delta_B \geq 1 \\ \delta_A, \delta_B \in \{0,1\} \end{array}$$

C)
$$\left\{ \begin{array}{l} x_2 \geq 0 \\ x_2 \leq 4 \\ 4x_1 - x_2 \geq 0 \\ x_1 \leq 2 \end{array} \right\} \cap \left\{ \begin{array}{l} x_2 \geq 0 \\ x_2 \leq 4 \\ 4x_1 + x_2 \leq 20 \\ x_1 \geq 3 \end{array} \right\}$$

D)
$$\begin{array}{l} x_2 \geq 0 \\ x_2 \leq 4 \\ 4x_1 - x_2 \geq 0 \\ 4x_1 + x_2 \leq 20 \\ x_1 \leq 2 \\ x_1 \geq 3 \\ x_2 \leq 2 \end{array}$$

E)
$$\left\{ \begin{array}{l} x_2 \geq 0 \\ x_2 \leq 4 \\ 4x_1 - x_2 \geq 0 \\ x_1 \leq 2 \end{array} \right\} \cup \left\{ \begin{array}{l} x_2 \geq 0 \\ x_2 \leq 4 \\ 4x_1 + x_2 \leq 20 \\ x_1 \geq 3 \end{array} \right\}$$

F)
$$\begin{array}{l} x_2 \geq 0 \\ x_2 \leq 4 \\ 4x_1 - x_2 \geq 0 \\ 4x_1 + x_2 \leq 20 \end{array}$$

Cutting the depots

The capacity of a depot used by Tuberg breweries is given by the following LP-model:

$$\begin{array}{llcllr} \text{maximize} & x_1 & + & 4x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq 4 & \text{(a)} \\ & x_1 & + & 5x_2 & \leq 15 & \text{(b)} \\ & x_1 & & & \leq 3 & \text{(c)} \\ & x_1, x_2 \geq 0 & & & & \end{array}$$

Q13: Solve the model to LP-optimality. What is the optimal solution?

A) $x_1 = 0, x_2 = 3$ B) $x_1 = 3, x_2 = 1$
C) $x_1 = 3, x_2 = 0$ D) $x_1 = \frac{5}{4}, x_2 = \frac{11}{4}$
E) $x_1 = 0, x_2 = 4$ F) $x_1 = 3, x_2 = \frac{13}{5}$

Q14: What is the corresponding dual solution?

A) $y_1 = \frac{1}{4}, y_2 = \frac{3}{4}, y_3 = 0$ B) $y_1 = 0, y_2 = \frac{3}{4}, y_3 = \frac{1}{4}$
C) $y_1 = \frac{3}{4}, y_2 = 0, y_3 = \frac{1}{4}$ D) $y_1 = 0, y_2 = 0, y_3 = \frac{49}{12}$
E) $y_1 = \frac{49}{16}, y_2 = 0, y_3 = 0$ F) $y_1 = 0, y_2 = \frac{4}{5}, y_3 = \frac{1}{12}$

Q15: The dual problem is to be solved to integer-optimality. The first equation in the simplex tableau is given by

$$y_1 + \frac{5}{4}y_3 - \frac{5}{4}y_4 + \frac{1}{4}y_5 = \frac{1}{4}$$

where $y_4, y_5 \geq 0$ are the slack variables in the dual problem. Derive a Gomory cut from the equation. Which inequality appears (after elimination of the slack variables)?

A) $2y_1 + y_2 \geq \frac{1}{4}$ B) $y_1 + 2y_2 \geq 4$
C) $y_1 - y_2 \geq 4$ D) $4y_1 + y_2 + y_3 \geq 1$
E) $y_1 + 2y_2 + y_3 \geq 2$ F) $y_1 - y_2 \geq 4$

Q16: Lagrangian relax constraint (b) in the primal model using multiplier $\lambda \geq 0$, and solve the corresponding Lagrangian dual. What is the optimal choice of Lagrangian multiplier λ?

A) $\lambda = \frac{1}{4}$ B) $\lambda = \frac{3}{4}$
C) $\lambda = \frac{5}{7}$ D) $\lambda = \frac{2}{3}$
E) $\lambda = 1$ F) $\lambda = \frac{4}{5}$

Cover inequalities

It is well-known that the mathematical symbol "{" (also known as a Tuberg) was invented by the Tuberg breweries long time before it became a standard symbol in mathematics. In a similar way, Cover inequalities were studied by the Tuberg designers more than 100 years ago, in the context of packing most possible beer bottles into a knapsack.

Q17: Consider the inequality

$$8x_1 + 8x_2 + 7x_3 + 5x_4 + 3x_5 + 2x_6 + 2x_7 \leq 11$$

Which of the following inequalities is not a minimal cover inequality?

A) $x_1 + x_2 \leq 1$ B) $x_1 + x_4 \leq 1$
C) $x_2 + x_5 + x_6 \leq 2$ D) $x_3 + x_5 + x_6 \leq 2$
E) $x_3 + x_4 + x_7 \leq 2$ F) $x_2 + x_4 \leq 1$

Q18: A valid inequality is

$$x_4 + x_5 + x_6 + x_7 \leq 3$$

What is the largest value of α such that the inequality

$$\alpha x_1 + x_4 + x_5 + x_6 + x_7 \leq 3$$

is valid?

A)	$\alpha = -1$	B)	$\alpha = 0$
C)	$\alpha = 1$	D)	$\alpha = 2$
E)	$\alpha = 3$	F)	$\alpha = 4$

In general we may let $N = \{1, \ldots, n\}$ and consider the knapsack polytope P given by

$$P = \text{conv}\left\{ x \in \mathbb{B}^n : \sum_{j \in N} a_j x_j \leq b \right\}$$

Assume that all $a_j \geq 0$, $j = 1, \ldots, n$ and $b \geq 0$.

Q19: (text question) Prove that the dimension of P is

$$\dim(P) = n - |B|$$

where $|B| = \{j \in N : a_j > b\}$.

Q20: (text question) Assume that $a_j \leq b$ for $j \in N$. Prove that for a given $j \in N$ the inequality

$$x_j \leq 1$$

is facet defining for P if and only if

$$a_j + \left(\max_{i \in N \setminus \{j\}} a_i \right) \leq b$$

THE END