Answers, Written Exam, 16 December 1999, David Pisinger

Planning the study

Answer 11 The correct formulation is

$$10\delta_A \leq x_A, \quad 15\delta_B \leq x_B, \quad 13\delta_C \leq x_C, \quad 8\delta_D \leq x_D, \quad 12\delta_E \leq x_E.$$

since this has the effect that if $\delta_i = 1$ then $x_i \geq m_i$ where m_i is the minimum hours of preparation for course i. Thus 11.b) is the correct answer. ■

Answer 12 Only one of the courses A and B can be followed

$$\delta_A + \delta_B \leq 1$$

That course B or C demands course D is written

$$\begin{array}{l} \delta_B \leq \delta_D \\ \delta_C \leq \delta_D \end{array}$$

To ensure that no more than 1100 kr are used for the books can be expressed as

$$30\delta_A + 20\delta_B + 80\delta_C + 40\delta_D + 45\delta_E \leq 110$$

If the student is using more than 25 hours on courses A,B,C he should also use at least 10 hours on courses D,E. For this purpose we introduce a new boolean variable δ which is 1 when $x_A + x_B + x_C \geq 25$. This leads to the following constraints

$$\begin{array}{l} x_A + x_B + x_C - 144\delta \leq 24 \\ x_D + x_E - 10\delta \geq 0 \end{array}$$

Thus the wrong constraint is

$$x_D + x_E \geq \frac{10}{25}(x_A + x_B + x_C)$$

For instance if $x_A + x_B + x_C = 50$ it will push $x_D + x_E \geq 20$ which was not the intention. Thus the correct answer is 12.e). ■

Answer 13 We use the algorithm from exercise 10. Since we have the solution

$$\delta_A = 0, \; \delta_B = 1, \; \delta_C = \frac{1}{16}, \; \delta_D = 1, \; \delta_E = 1$$

the correct formulation is

$$\begin{array}{rl} \gamma = \min & 1x'_A + 0x'_B + \frac{15}{16}x'_C + 0x'_D + 0x'_E \\ \text{s.t.} & 30x'_A + 20x'_B + 80x'_C + 40x'_D + 45x'_E \geq 111 \\ & x'_A, x'_B, x'_C, x'_D, x'_E \in \{0,1\} \end{array}$$

thus 13.c) is correct. ■

Answer 14 The solution to the above knapsack problem is easily found by inspection. Since we should choose the solution with most possible items chosen, we set $x'_B = x'_D = x'_E = 1$, which gives the weight sum 105 in the capacity constraint. To pass 111 we may choose either $x'_A = 1$ or $x'_C = 1$, where $x'_C = 1$ is the choice giving the smallest objective. Thus we have the solution

$$x'_B = x'_C = x'_D = x'_E = 1$$

with objective value $\gamma = \frac{15}{16} < 1$. We may derive the cover inequality

$$\delta_B = \delta_C = \delta_D = \delta_E \leq 4 - 1 = 3$$

and 14.d) is correct.

Solving the problem to LP-optimality with the new constraint added, we find the solution

$$\delta_A = \frac{1}{18}, \; \delta_B = \frac{17}{18}, \; \delta_C = \frac{1}{18}, \; \delta_D = 1, \; \delta_E = 1$$

with objective value 3.055. Without the new constraint we had the objective value 3.063, thus the constraint did have an effect. The integer-optimal solution is

$$\delta_B = 1, \; \delta_D = 1, \; \delta_E = 1, \; x_B = 15, \; x_D = 8, \; x_E = 12.$$

■

Pasta production

Answer 15 If the matrix A is totally unimodular (TU), we will obtain a basic integer solution x to the problem for any choice of c and b as long as b is integral. It is easily seen that A_1 is not TU, since column 2 and 4 contain the sub-matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ which has determinant -2. The same applies for matrix A_2 where column 4 and 5 contain the sub-matrix $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ with determinant 2. Matrix A_3 contains the sub-matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ with determinant 2. None of A_5 and A_6 satisfy the property that all entries are in {0,1,-1}.

To prove that A_4 is TU, we prove that it satisfies property P (Williams page 208). We divide the columns in two sets: columns $\{1,2,3\}$ and $\{4,5\}$. Every row contains exactly two entries different from zero, and if they have the same sign, they are in different sets. Thus the correct answer is 15.d). ■

Gomory cut

Answer 16 From the objective function it is seen that x_3 and x_4 are not in the basis. The remaining variables are in diagonal form, and all coefficients to nonbasis variables in the objective function are negative. Thus we know that the algorithm has terminated. The correct solution is read as

$$x_1 = \frac{20}{7}, \; x_2 = 3, \; x_5 = \frac{23}{7}$$

Thus answer 16.a) is correct. ■

Answer 17 The first constraint reads

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7}$$

which leads to the Gomory cut

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

which by multiplication with 7 gives 17.e).

Adding the inequality to the problem and solving the model to LP-optimality, we find the solution

$$x_1 = 2, \; x_2 = \frac{1}{2}, \; x_3 = 1, \; x_4 = \frac{5}{2}$$

with objective value 7.5. The original problem had the solution value 8.429 so the cut had a considerable effect. The integer-optimal solution is

$$x_1 = 2,\ x_2 = 1,\ x_3 = 2,\ x_4 = 2,\ x_5 = 1$$

with objective value 7. ■

Modular arithmetics

Answer 18 We have the equation

$$21x_1 + 43x_2 + 7x_3 = 35$$

taking the remainder modulo 6, we get the inequality

$$3x_1 + x_2 + x_3 \geq 5$$

Thus answer 18.b) is correct. ■

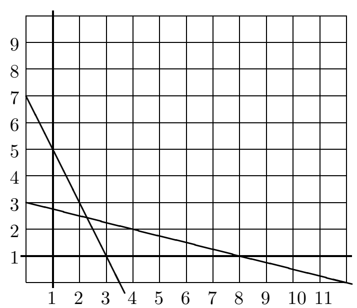
Spaghetti Bolognese

Answer 19 Lagrangian relaxing the first constraint we get the problem

$$\begin{array}{rrcrrr} z = \max & (2-\lambda)x_1 & + & (7-4\lambda)x_2 & +12\lambda & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq & 7 \\ & & & x_2 & \geq & 1 \\ & x_1 & & & \geq & 1 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0,\ \text{integer}} \end{array}$$

■

Answer 20 By drawing the problem we see that the remaining constraints define the convex hull. In this case we know that the best choice of the lagrangian multiplier corresponds to the dual variable of the relaxed constraint.



The LP-optimal solution of the original problem is

$$x_1 = \frac{16}{7},\ x_2 = \frac{17}{7}$$

which can be solved graphically. The corresponding objective value is $\frac{151}{7}$. To find the dual variables we use complementary slackness. The dual problem is defined as

$$\begin{array}{rrcrcrcrcl} z = \min & 12y_1 & + & 7y_2 & - & y_3 & - & y_4 & & \\ \text{s.t.} & y_1 & + & 2y_2 & & & - & y_4 & \geq & 2 \\ & 4y_1 & + & y_2 & - & y_3 & & & \geq & 7 \\ & \multicolumn{9}{l}{x_1, x_2 \geq 0, \text{integer}} \end{array}$$

Since both of the primal variables $x_1, x_2 > 0$ the two constraints are binding thus we have

$$\begin{array}{rcrcrcrcl} y_1 & + & 2y_2 & & & - & y_4 & = & 2 \\ 4y_1 & + & y_2 & - & y_3 & & & = & 7 \end{array}$$

We know that $y_3 = y_4 = 0$, since the LP-optimum of the primal problem is not binding for the two last constraints. This melts down to the equations

$$\begin{array}{rcrcl} y_1 & + & 2y_2 & = & 2 \\ 4y_1 & + & y_2 & = & 7 \end{array}$$

with optimal solution

$$y_1 = \frac{12}{7},\ y_2 = \frac{1}{7}$$

The optimal choice of λ is $\lambda = y_1 = \frac{12}{7}$. In this case we find that the solution to the lagrangian relaxed problem is $x_1 = 3$, and $x_2 = 1$. The objective then becomes $\frac{151}{7}$, which is equivalent to the LP-solution to the original problem. ■