

Answers, Written Exam, 17 December 1998, David Pisinger

Questions Q11-Q13: Planning pension

Answer 11 The optimal solution can be found from the dual solution through complementary slackness. In our case, we have five proposals given, and thus it is much easier to verify a solution. The optimal solution must satisfy:

- The primal solution should be feasible.
- The maximum objective value $x_1 + x_2 + x_3$ equals the dual solution value $6y_1 + 8y_2 + y_3 = 6 \cdot \frac{2}{9} + 8 \cdot 0 + \frac{5}{9} = \frac{17}{9}$.

We have: 11.a) does not obtain the correct objective value, 11.b) has the correct objective value, but it is not feasible (last constraint is violated), 11.c) has a too large objective value, 11.d) has the correct objective value and is feasible, 11.e) does not obtain the correct objective value. Thus answer 11.d) is the correct one. ■

Answer 12 Using the formula from Nemhauser and Wolsey page 212, we have the equation

$$x_1 + \frac{11}{9}x_2 + \frac{1}{9}s_1 + \frac{7}{9}s_3 = \frac{13}{9}$$

which gives the Gomory cutting plane

$$\frac{2}{9}x_2 + \frac{1}{9}s_1 + \frac{7}{9}s_3 \geq \frac{4}{9}$$

Substituting the proper values of s_1 and s_2 one gets

$$x_1 + x_2 \leq 1$$

Thus 12.b) is correct. Adding this constraint to the model, one gets the objective 1.57143 which is considerably tighter than the original LP-solution of 1.88889. ■

Answer 13 Using multipliers (2, 1, 3) for the three constraints and adding them together, one gets:

$$8x_1 + 8x_2 + 12x_3 \leq 23$$

dividing by four and rounding down, one gets

$$2x_1 + 2x_2 + 3x_3 \leq \left\lfloor \frac{23}{4} \right\rfloor = 5$$

Thus 13.d) is correct. Adding this constraint to the model as well as the previous Gomory cuts, one gets the objective 1.57143, which did not improve the objective. ■

Questions Q14-Q16: Still active

Answer 14 The proper formulation is: The professor may work at most 37 hours a week:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 37$$

We introduce binary decision variables δ_i which attain the value $\delta_i = 1$ iff the professor becomes engaged in project i . Since the professor will work at least 5 hours, if he starts working on a project, we may choose $\epsilon = 5$. An upper bound M on x_i is 37.

$$x_i - 5\delta_i \geq 0 \text{ for } i = 1, \dots, 5$$

$$x_i - 37\delta_i \leq 0 \text{ for } i = 1, \dots, 5$$

Only one of the projects 2 and 3 can be started:

$$\delta_2 + \delta_3 \leq 1$$

If project 1 is started, then at least 20 hours should be used on projects 3 and 4. More formally this means $\delta_1 = 1 \Rightarrow x_3 + x_4 \geq 20$. A lower bound m on $x_3 + x_4 - 20$ is $m = -20$, thus we get the formulation

$$x_3 + x_4 - 20\delta_1 \geq 0$$

If at least 25 hours a week are used on projects 1, 3, 4, 5 then at least 5 hours should be used on project 2. Since the minimal working effort on a project is 5 hours, this is equivalent to saying $x_1 + x_3 + x_4 + x_5 \geq 25 \Rightarrow \delta_2 = 1$. An upper bound M on $x_1 + x_3 + x_4 + x_5 - 25$ is $37 - 25 = 12$. As stated in the question we may set $\epsilon = 1$ thus getting

$$x_1 + x_3 + x_4 + x_5 - 13\delta_2 \leq 24$$

All δ_i are binary variables:

$$\delta_i \in \{0, 1\} \text{ for } i = 1, \dots, 5$$

Now to the *wrong* part: The inequality 14.f) has the form

$$x_1 + x_3 + x_4 + x_5 - 26\delta_2 \geq 26$$

which says $x_1 + x_3 + x_4 + x_5 \leq 25 \Rightarrow \delta_2 = 1$. This is exactly the reverse of what the professor demanded, so 14.f) is wrong. ■

Answer 15 If we denote the indices by a_i we have an inequality of the form

$$a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_4 + a_5\delta_5 \leq 20$$

In 15.c) the sum of $a_3 + a_4 + a_5 = 4 + 9 + 6$ does not exceed 20 and is accordingly not a cover. In particular, it is not a *minimal* cover. ■

Answer 16 The indices $\{1, 2, 3, 5\}$ can be extended to a cover $\{1, 2, 3, 4, 5\}$ since $a_4 \geq a_i$ for $i = 1, 2, 3, 5$. In this way we get the cover inequality

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 \leq 3$$

Thus the correct answer is $m = 3$ as stated in 16.c). ■

Questions Q17-Q20: Touring Medal

Answer 17 One could simply test which of the given solutions is a correct dual solution. But the correct way of solving the problem is to use complementary slackness.

The first inequality is tight since $x_1 - x_2 = 3$ so the dual variable y_1 may attain a nonnegative value. The third inequality is also tight, since $x_1 + x_2 + x_3 = 8$, so the same applies for y_3 . All dual variables, corresponding to constraints which are not binding, have the value zero, i.e. $y_2 = y_4 = y_5 = 0$. Using the arguments in the reverse form for the primal variables, we know that $x_1 \neq 0$ and $x_2 \neq 0$. Looking at the dual constraints, one must have $y_1 + y_3 + y_4 = 1$ and $-y_1 + y_2 + y_3 + y_5 = 1$. Since $y_2 = y_4 = y_5 = 0$ we can immediately see that $y_1 = 0$ and $y_3 = 1$. Thus the correct answer is 17.a). ■

Answer 18 If we remove the third constraint $x_1 + x_2 + x_3 \leq 8$ then the matrix corresponding to the left-hand side of the problem, satisfies property P: Every row has at most two elements. All elements are 0, 1, -1. Moreover, if we assign all columns to one class, then we satisfy the last criteria: If a row contains two non-zero elements of different sign, then they belong to the same class. Thus the correct answer is 18.c). ■

Answer 19 Lagrangian relaxing the constraint $x_1 + x_2 + x_3 \leq 8$ we get the objective function

$$x_1 + x_2 + x_3 - x_4 - x_5 - 2(x_1 + x_2 + x_3 - 8)$$

which gives

$$-x_1 - x_2 - x_3 - x_4 - x_5 + 16$$

Thus the correct answer is 19.d). ■

Answer 20 If the remaining problem is totally unimodular after having Lagrangian relaxed some constraints, then the best choice of λ is to choose the dual variables associated with the relaxed constraints. From the previous questions we know that $y_3 = 1$ thus choosing $\lambda = 1$ leads to the tightest bound, and 20.b) is correct. ■