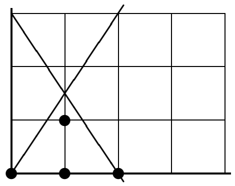
Answers, Written Exam, 15 December 2000, David Pisinger

Answer 11 Drawing the constraints and the integer points in the set, one gets



It is easy to see that (i) and (ii) are certaintly not facet defining, while constraint (iv) is facet defining. Finally (iii) is redundant, since it could be removed without changing the solution set. Thus the correct answer is 11.d). ■

Answer 12 We add the slack variables getting the tableau

$$\begin{aligned} z &= x_2 \\ x_3 &= 6 - 3x_1 - 2x_2 \\ x_4 &= 0 + 3x_1 - 2x_2 \end{aligned}$$

After two pivot operations we get

$$\begin{aligned} z &= \tfrac{3}{2} - \tfrac{1}{4}x_3 - \tfrac{1}{4}x_4 \\ x_1 &= 1 - \tfrac{1}{6}x_3 + \tfrac{1}{6}x_4 \\ x_2 &= \tfrac{3}{2} - \tfrac{1}{4}x_3 - \tfrac{1}{4}x_4 \end{aligned}$$

Thus answer 12.a) is correct. One could also notice that sinze $z = x_2$ the two expressions defining z and x_2 must be equal, and thus only 12.a) can be correct. ■

Answer 13 From the equation $z + \frac{1}{4}x_3 + \frac{1}{4}x_4 = \frac{3}{2}$ we get the Gomory cut as described in Wolsey, Section 8.6. This gives

$$\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

Multiplying by 4 we get $x_3 + x_4 \geq 2$, thus answer 13.e) is correct.

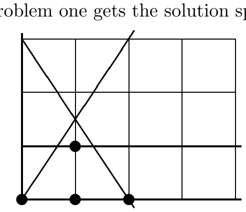
Introducing the original variables in place of $x_3 = 6 - 3x_1 - 2x_2$ and $x_4 = 3x_1 - 2x_2$ we get

$$(6 - 3x_1 - 2x_2) + (3x_1 - 2x_2) \geq 2$$

which may be reduced to

$$x_2 \leq 1$$

Adding the inequality to the problem one gets the solution space



The LP-optimal solution now becomes $x_1 = \frac{4}{3}$ and $x_2 = 1$ with objective value $z = 1$. ■

Answer 14 Lagrangian relaxing the second constraint $-3x_1 + 2x_2 \leq 0$ in (1) using a multiplier $\lambda \geq 0$ gives the objective function

$$x_2 - \lambda(-3x_1 + 2x_2 - 0) = x_2 + 3\lambda x_1 - 2\lambda x_2 = 3\lambda x_1 + (1 - 2\lambda)x_2$$

Thus answer 14.d) is correct. ■

Answer 15 The remaining constraints define the set

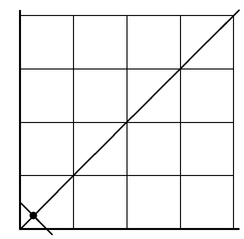
$$X = \left\{ \begin{array}{l} 3x_1 + 2x_2 \leq 6 \\ x_1, x_2 \geq 0, \text{ integer} \end{array} \right\}$$

where we notice that $\text{conv}(X) = X$. In this situation Corrolary page 173 in Wolsey say that the best lagrangian relaxation is as strong as the LP-relaxation. Thus we know that the best Lagrangian multiplier λ corresponds to the dual variable of the original problem (1) solved as an LP-problem.

The dual problem is

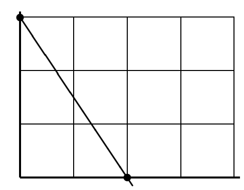
$$\begin{array}{lrcrcl} \text{minimize} & 6y_1 & & & & \\ \text{subject to} & 3y_1 & - & 3y_2 & \geq & 0 \\ & 2y_1 & + & 2y_2 & \geq & 1 \\ & y_1, y_2 \geq 0 & & & & \end{array}$$

where the LP-optimal solution may be found graphically as $y_1 = y_2 = \frac{1}{4}$.



Thus the optimal choice of the Lagrangian multiplier is $\lambda = y_2 = \frac{1}{4}$, making answer 15.e) correct.

One could also try all the proposed values of λ and see which of them results in the tightest bound. This can be done graphically as the set of feasible solutions is independant of the value of λ:



For each of the values of λ we find

$$\begin{array}{llll} \lambda = 0: & z = x_2, & x^* = (0,3), & z^* = 3 \\ \lambda = 1: & z = 3x_1 - x_2, & x^* = (2,0), & z^* = 6 \\ \lambda = 2: & z = 6x_1 - 3x_2, & x^* = (2,0), & z^* = 12 \\ \lambda = \frac{1}{5}: & z = \frac{3}{5}x_1 + \frac{2}{5}x_2, & x^* = (0,3), & z^* = \frac{6}{5} \\ \lambda = \frac{1}{4}: & z = \frac{3}{4}x_1 + \frac{1}{2}x_2, & x^* = (2,0) \vee (0,3), & z^* = \frac{3}{2} \\ \lambda = \frac{1}{3}: & z = x_1 + \frac{1}{3}x_2, & x^* = (2,0), & z^* = 2 \end{array}$$

■

Answer 16 Using complementary slackness we know that the dual constraint is zero if and only if the constraint is not binding. Since $y_4 \neq 0$ constraint 4 must be binding, thus $x_2 = 1$. This leaves only one possible answer 16.a). ■

Answer 17 To separate the most violated cover inequality from

$$5x_1 - 6x_2 + 5x_3 + 8x_4 \leq 5$$

we use the procedure described in Wolsey page 150. Thus first we need to ensure that all coefficients are positive by substituting $x_2' = 1 - x_2$, getting

$$5x_1 + 6x_2' + 5x_3 + 8x_4 \leq 11$$

we solve the separation problem

$$\begin{aligned} \gamma = \text{minimize} \quad & \sum_{i \in I} (1 - x_i)\delta_i \\ \text{subject to} \quad & \sum_{i \in I} a_i \delta_i \geq b + 1 \\ & \delta_i \in \{0, 1\}, \; i \in I. \end{aligned}$$

In the current solution we have $x_2 = 1$ and thus $x_2' = 0$ meaning that the separation problem becomes

$$\begin{aligned} \text{minimize} \quad & 1\delta_1 + 1\delta_2 + \tfrac{2}{17}\delta_3 + \tfrac{3}{17}\delta_4 \\ \text{subject to} \quad & 5\delta_1 + 6\delta_2 + 5\delta_3 + 8\delta_4 \geq 12 \\ & \delta_i \in \{0, 1\}, \; i = 1, \ldots, 4 \end{aligned}$$

with optimal solution $\delta_1 = 0$, $\delta_2 = 0$, $\delta_3 = 1$, $\delta_4 = 1$. As the objective function is $\gamma = \frac{5}{17} < 1$ we have separated a cover inequality with cover $C = \{3, 4\}$. The inequality is

$$x_3 + x_4 \leq |C| - 1 = 1$$

Thus 17.d) is correct.

Adding the new inequality to the problem we get the solution value $z = \frac{17}{7} = 2.429$ The original solution value was $z = \frac{46}{17} = 2.706$, so the cut did have a considerable effect. ■

Answer 18 Using the algorithm described page 149 in Wolsey, we solve the knapsack problem

$$\begin{aligned} \gamma = \text{maximize} \quad & x_2 + x_3 + x_4 \\ \text{subject to} \quad & 3x_2 + 4x_3 + 3x_4 \leq 9 - 7 \\ & x_i \in \{0, 1\}, \; i = 1, \ldots, 4 \end{aligned}$$

with optimal solution $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ and objective value $\gamma = 0$. This means that we may set $\alpha = |C| - 1 - \gamma = 3 - 1 - 0 = 2$. The correct answer is 18.b).

Adding the new inequality to the problem together with the previous cover inequality we get the solution value $z = 2$ and integer solutions $x_2 = x_3 = 1$. ■

Answer 19 We introduce the variables x_a, x_b, x_c to denote the amount of gifts of type A, B and C respectively. Moreover, we use s_a, s_b, s_c to denote the number of sleighs needed for transporting each

of the gift types. This immediately gives the formulation

$$\begin{aligned} \text{minimize} \quad & s_a + s_b + s_c \\ \text{subject to} \quad & x_a + x_b + x_c = 1000 \\ & 100s_a - x_a \geq 0 \\ & 200s_b - x_b \geq 0 \\ & 300s_c - x_c \geq 0 \\ & x_a - x_c \leq 100 \\ & x_c - x_a \leq 100 \\ & x_b - 501\delta \leq 499 \\ & x_a - 100\delta \geq 0 \\ & x_a, x_b, x_c \geq 0 \\ & s_a, s_b, s_c \geq 0, \text{ integer} \\ & \delta \in \{0, 1\} \end{aligned}$$

The first constraint ensures that 1000 gifts will be brought out. The three following constraints binds the number of sleighs to the number of gifts for each of the three gift types. The next two constraints ensure that the amount of gifts for type A and C do not differ by more than 100. Finally the last inequalities ensure that if $x_b \geq 500$ then $\delta = 1$, and again if $\delta = 1$ then $x_a \geq 100$.

The optimal solution (which was not asked for) is

$$s_a = 2, s_b = 3, s_c = 1, x_a = 150, x_b = 600, x_c = 250, \delta = 1$$

thus six sleighs are needed. ■

Answer 20 Assuming that the graph is $G = (V, E)$ the problem may be formulated as

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{i \in V} x_i \\ \text{subject to} \quad & x_i + x_j \geq 1 \quad (i, j) \in E \\ & x_i \in \{0, 1\} \quad i \in V \end{aligned}$$

Consider a triangle in the graph, e.g. spanned by the nodes $(1, 3, 6)$. Then we have the inequalities from the formulation

$$\begin{array}{ccccccc} x_1 & + & x_3 & & & \geq & 1 \\ & & x_3 & + & x_6 & \geq & 1 \\ x_1 & & & + & x_6 & \geq & 1 \end{array}$$

using the multipliers $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ for the three inequalities we get the new inequality

$$x_1 + x_3 + x_6 \geq \frac{3}{2}$$

rounding up the right-hand-side gives $x_1 + x_3 + x_6 \geq 2$.

Now, using this inequality for triangles $(1, 3, 6)$, $(2, 4, 7)$, $(3, 5, 1)$, $(4, 6, 2)$, $(5, 7, 3)$, $(6, 1, 4)$ and $(7, 2, 5)$, using multipliers $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ we get the new inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq \frac{14}{3}$$

rounding up the right-hand-side gives the stated. ■