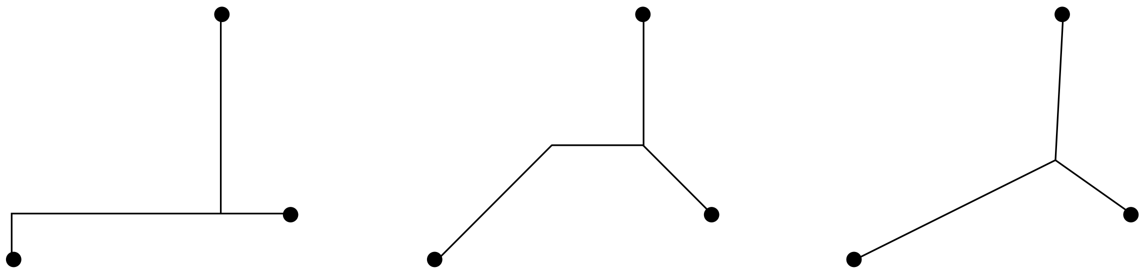
Flexibility of Steiner Trees in Uniform Orientation Metrics



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Outline

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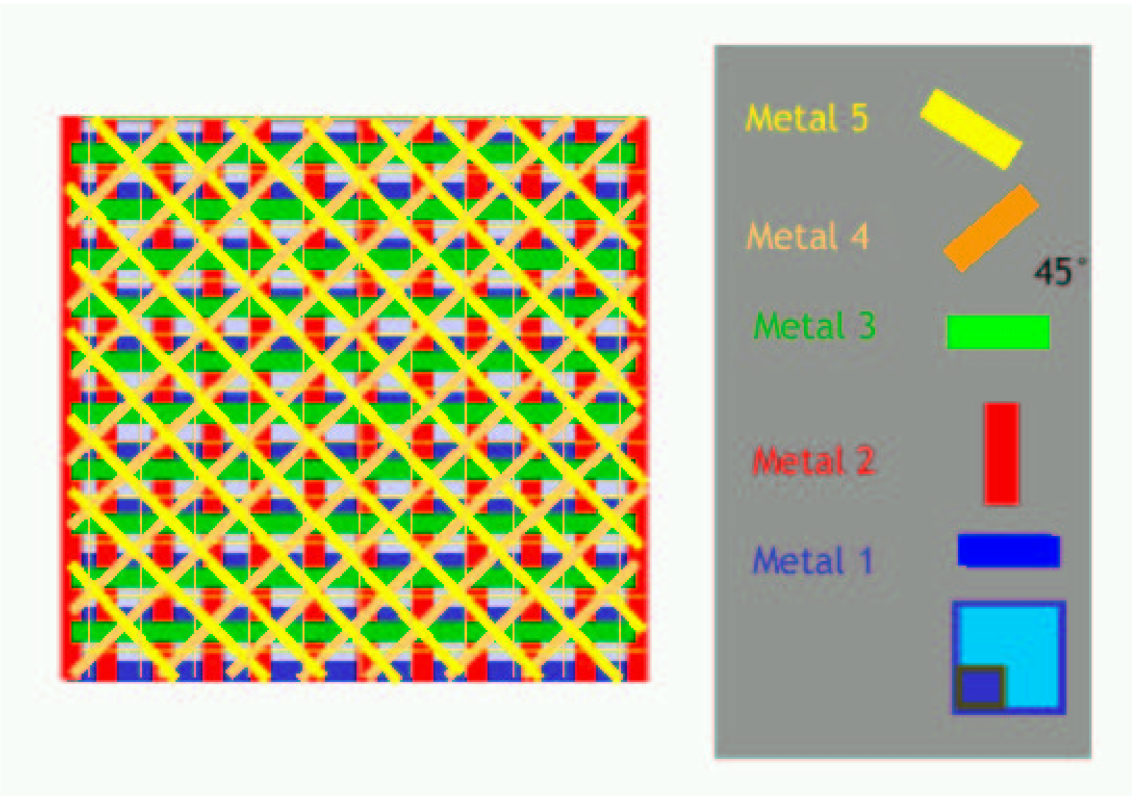
- Fixed and Uniform Orientation Metrics
- Steiner Trees in Uniform Orientation Metrics
- Edge Directions
- Length Preserving Shifts
- Steiner Trees for a Given Topology
- Flexibility Polygon

Motivation and Background

Motivation: Routing in VLSI design

- Manhattan architecture:
  Horizontal and vertical wires only

- X architecture:
  Horizontal, vertical and diagonal wires



X Architecture

Introduced in June 2001 by a consortium of chip manufacturers and chip software companies
$\longrightarrow$ `www.xinitiative.org`

Promises 20% reductions in wire length as a result of the use of diagonal routing — however, this requires placement and routing algorithms that take full advantage of the X architecture.

> "On every meaningful measure of layout quality, the X architecture is superior to the Manhattan architecture, which is why we expect that five years from today, virtually all high-end chips will use X."
> [Teig, 2002]

Fixed Orientation Metrics

Introduced by [Widmayer, Wu & Wong, 1987].

Given a set A of at least two distinct orientations in the plane.

Orientation = angle with the x-axis of corresponding straight line.

A-oriented line segment/line: The orientation of the segment/line is in A.

A-distance: Euclidean length of a shortest (zig-zag) path in which each line segment is A-oriented.

- $|pq|$: Euclidean distance between p and q
- $|pq|_A$: A-distance between p and q

Fixed Orientation Metrics: Basic Results

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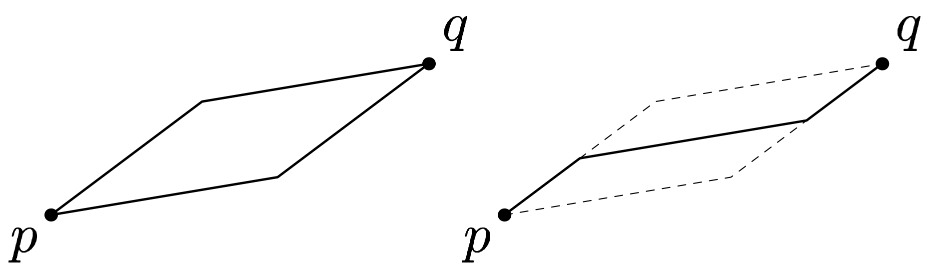
1. The A-distance induces a metric for any given set A.

2. For any two points p and q does there exist a point r such that

$$|pq|_A = |pr| + |rq|$$



3. For any A does there exist a constant c_A such that

$$|pq|_A \leq c_A |pq|$$

Uniform Orientation Metrics (λ-Metrics)

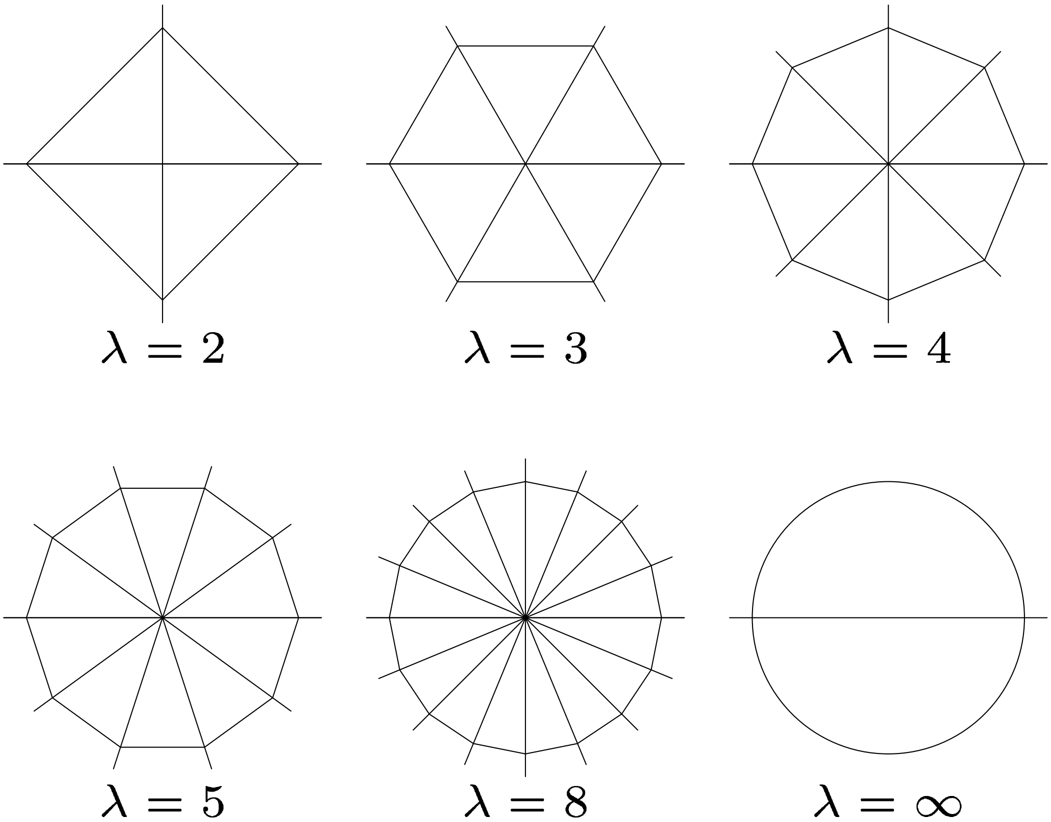
For $\lambda \geq 2$ consider fixed orientation metric given by the orientations

$$A_\lambda = \{\omega, 2\omega, \ldots, \lambda\omega\}$$

where $\omega = \pi/\lambda$.

λ-oriented line segment/line: The orientation of the segment/line is in A_λ.

λ-distance = A_λ-distance = Euclidean length of a shortest (zig-zag) path in which each line segment is λ-oriented.



Steiner Trees in Uniform Orientation Metrics (λ-SMTs)

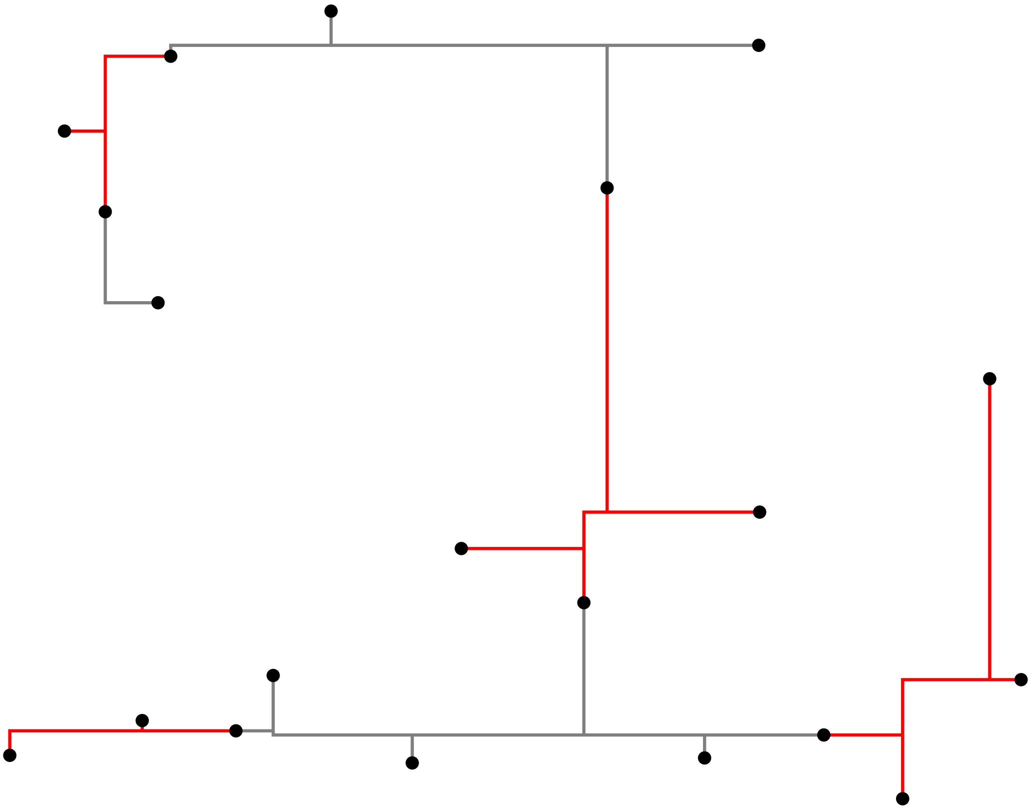
Definition: Given $\lambda \geq 2$ and a set N of terminals in the plane, find a shortest interconnection of the terminals under the λ-metric.

Complexity: NP-hard since the rectilinear Steiner tree problem ($\lambda = 2$) is a special case [Garey & Johnson, 1977].

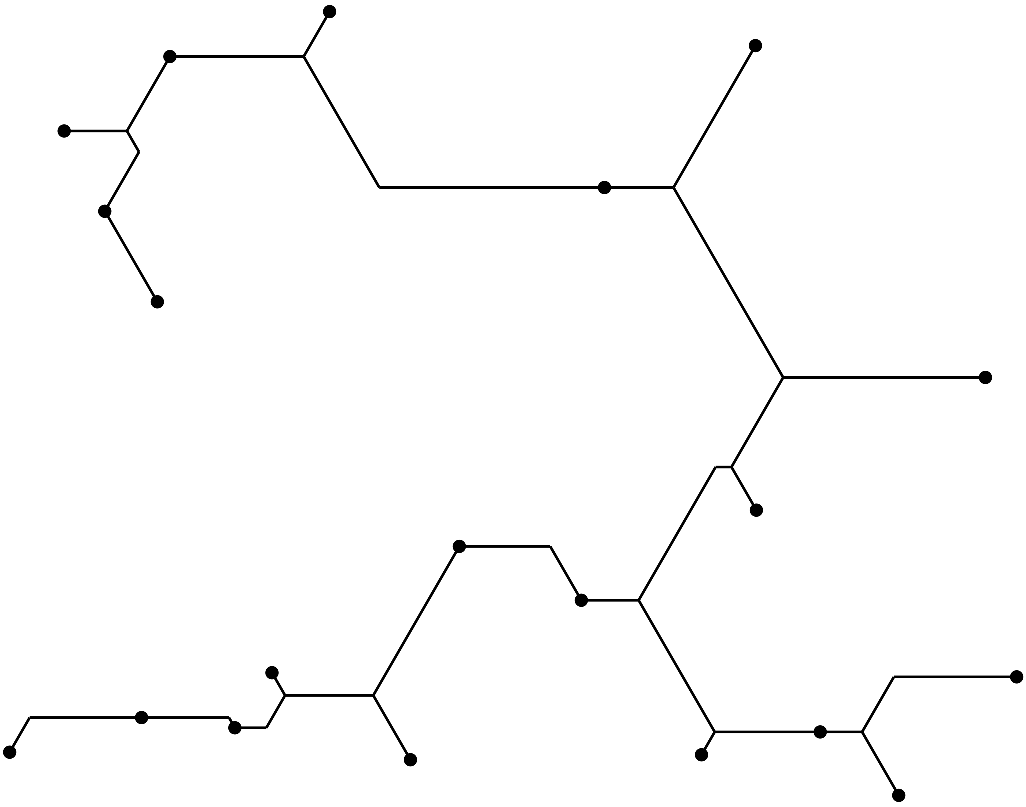
Approximation: Has a polynomial-time approximation scheme [Arora, 1996]; works since the λ-distance is within a constant factor of the Euclidean distance.

Applications: VLSI design, printed circuit board layout etc.

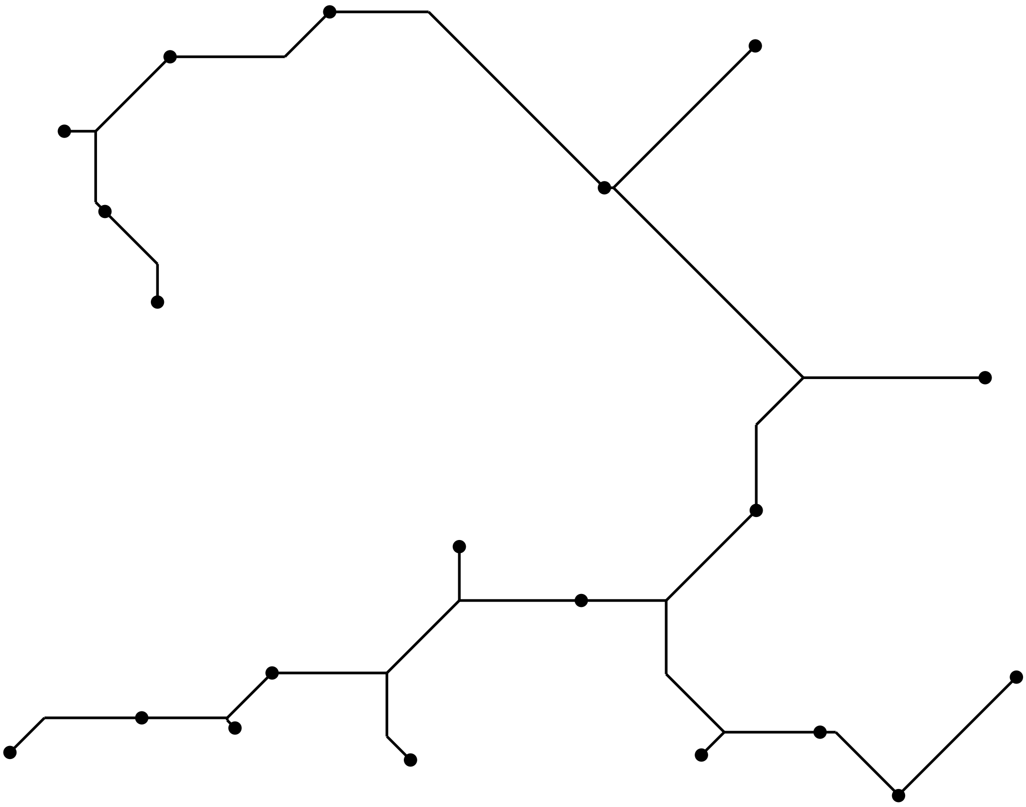
Rectilinear Steiner Tree ($\lambda = 2$)



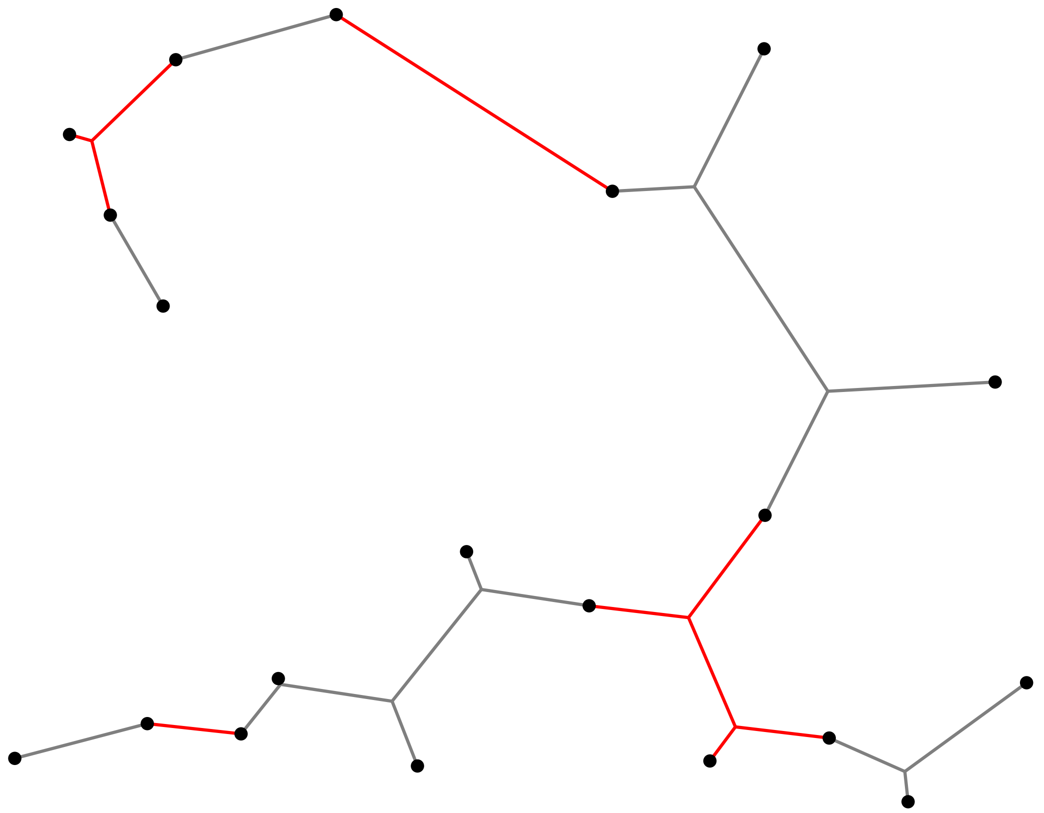
Hexagonal Steiner Tree ($\lambda = 3$)



Octilinear Steiner Tree ($\lambda = 4$)



Euclidean Steiner Tree ($\lambda = \infty$)



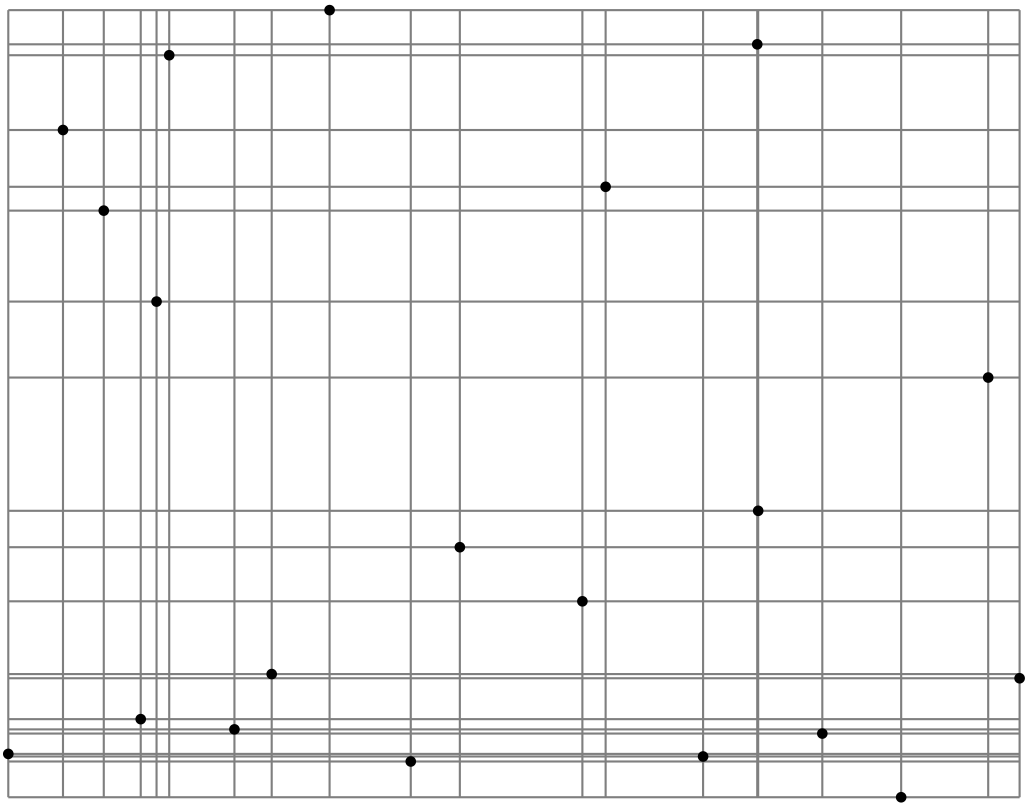
Full Steiner Trees and Fulsome λ-SMTs

Full Steiner tree (FST): Steiner tree in which all terminals are leaves (and all Steiner points are interior nodes). A λ-SMT is a union of FSTs.

Fulsome λ-SMT: A λ-SMT for which the number of FSTs is maximized. In particular, no FST can be split into two or more FSTs.

Canonical λ-SMT: Any characterization of λ-SMTs which reduces the set of optimal solutions; for $\lambda = 2$ one canonical form is the Hwang FST topology.

The Hanan Grid



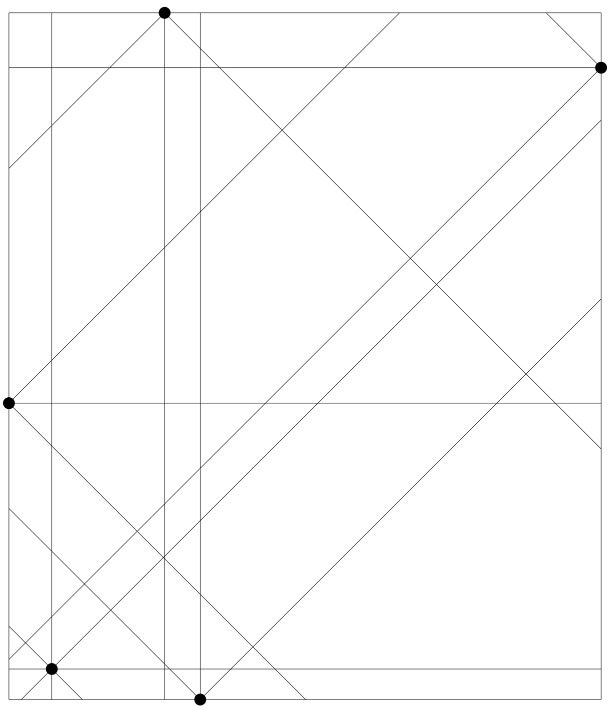
Theorem [Hanan, 1966]
There exists a rectilinear SMT for which all Steiner points are vertices in the grid graph for N.

Multi-level Grids

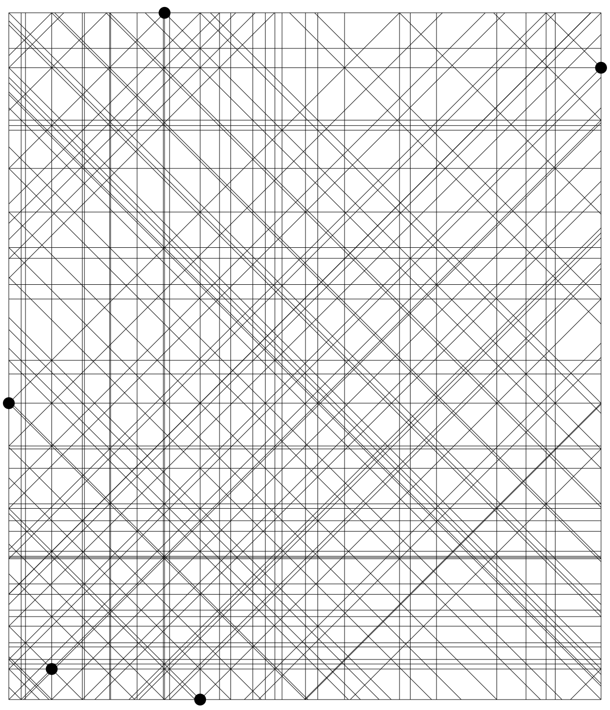
For a given set A of orientations recursively define

$GG_0(N) = N$

$i \geq 1$: $GG_i(N)$ = Intersections of all A-oriented lines through all points in $GG_{i-1}(N)$.



$GG_1(N)$



$GG_2(N)$

Note that for $\lambda = 2$ we have that $GG_1(N)$ is identical to the intersections of the Hanan grid for N.

Basic Structural Properties

Theorem [Brazil, Thomas & Weng, 2000]
The minimum angle at a Steiner point is $\lceil 2\lambda/3 - 1 \rceil \omega$ while the maximum angle is $\lfloor 2\lambda/3 + 1 \rfloor \omega$.
(Note that $(2\lambda/3)\omega = 2\pi/3 = 120°$).

Corollary
The maximum degree of any vertex in a λ-SMT is 3, except when $\lambda = 2, 3, 4$ or 6.

Theorem [Brazil, Thomas & Weng, 2000]
For every set N of terminals there exists a λ-SMT for N such that each of its FSTs has at most one bent edge.

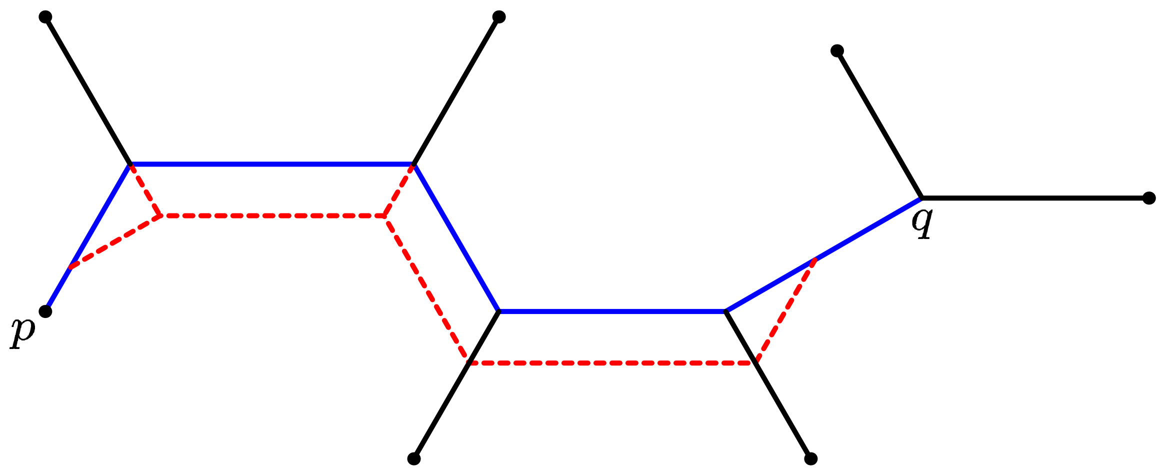
Theorem [Brazil, Thomas & Weng, 2000]
For each set N of n terminals, there exists a λ-SMT T for N such that all Steiner points in T are grid points in $GG_{n-2}(N)$.

Shifts in λ-Geometry

Given two nodes p and q in a tree T, a shift is a perturbation of the internal Steiner points on the path P from p to q in T, such that

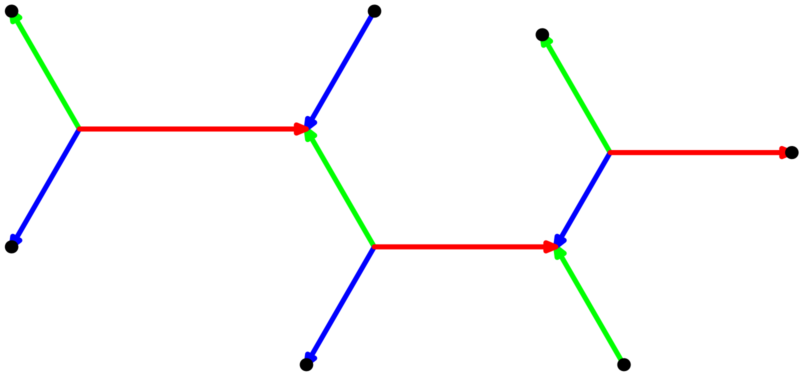
(1) each Steiner point moves along the incident edge which is not on P

(2) each internal edge is keeps its orientation

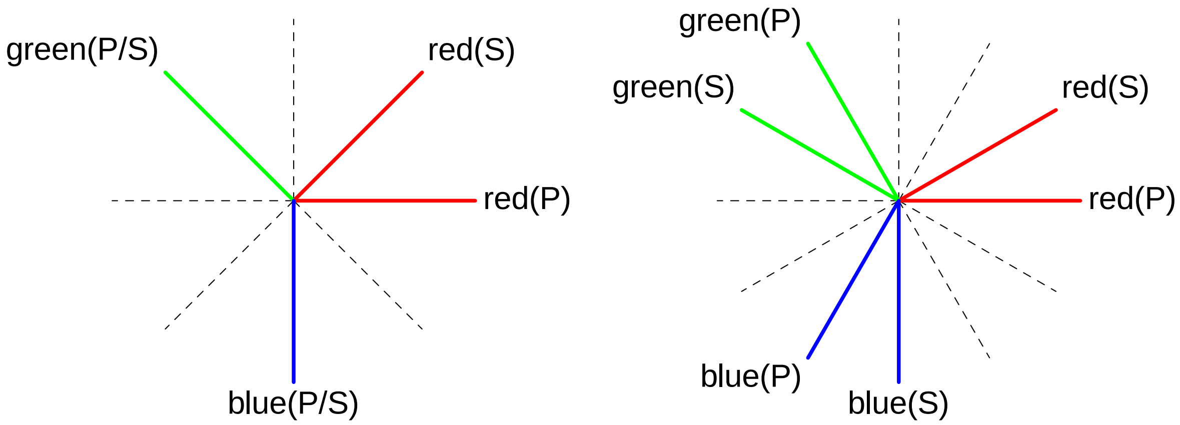


Edge Coloring and Primary/Secondary Labeling

Direct and color edges of a full λ-SMT:



Theorem
The edges in a full λ-SMT can have at most 4 different directions for $\lambda \neq 3m$ and at most 6 different directions for $\lambda = 3m$ (called direction sets).



$\lambda = 4$ $\qquad$ $\lambda = 6$

Identify (exclusively) primary and secondary edges.

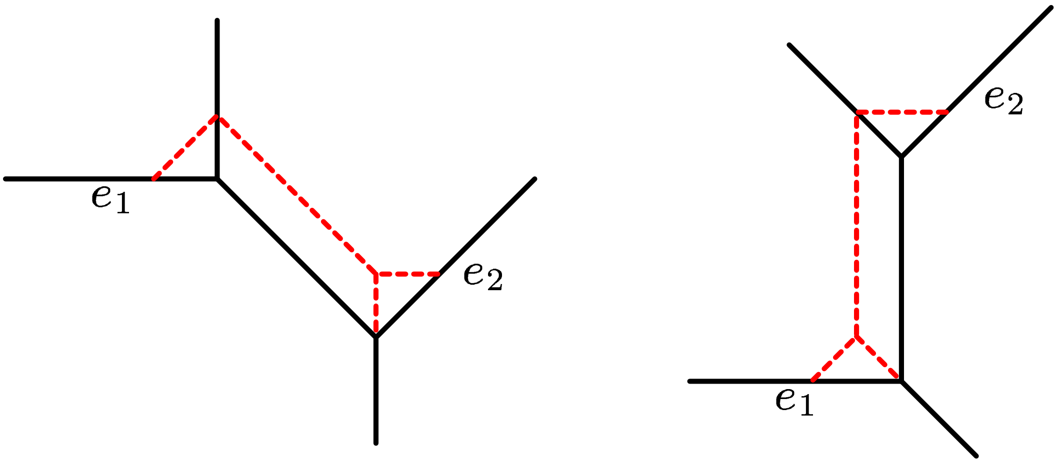
Zero-Shifts

A zero-shift in a full λ-SMT T is a shift that does not increase the length of T.

Theorem
Given an exclusively primary edge or half-edge e_1 and an exclusively secondary edge or half-edge e_2 in a full λ-SMT, there exists a zero-shift on the path P between e_1 and e_2.

Proof
By induction on the number of internal edges on the path P.

Basis (one internal edge):



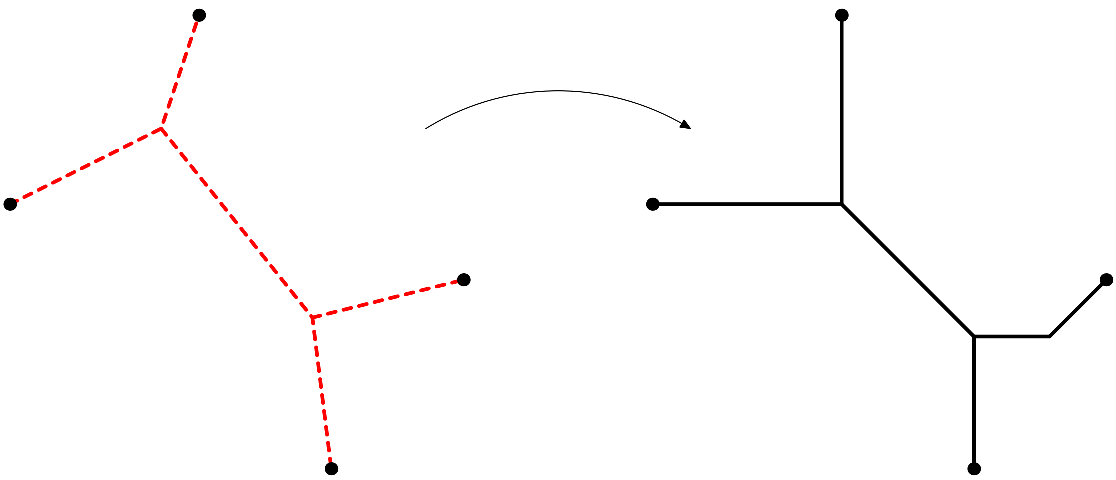
Inductive step: Decompose P into two subpaths and move primary/secondary material to intermediate half edge or exclusively primary/secondary edge.

Steiner Trees for a Given Topology

Topology: Graph structure for the interconnections of terminals and/or Steiner points.

Full Steiner topology: A topology for which all terminals have degree 1 (are leaves) and all Steiner points have degree 3.

Given a λ-metric and a full Steiner topology $\mathcal{T}$ for a set N of n terminals, find a λ-SMT for $\mathcal{T}$ (or locate the Steiner points in $\mathcal{T}$ such that the corresponding tree is a local minimum).



Topology $\mathcal{T}$ $\qquad$ λ-SMT for $\mathcal{T}$

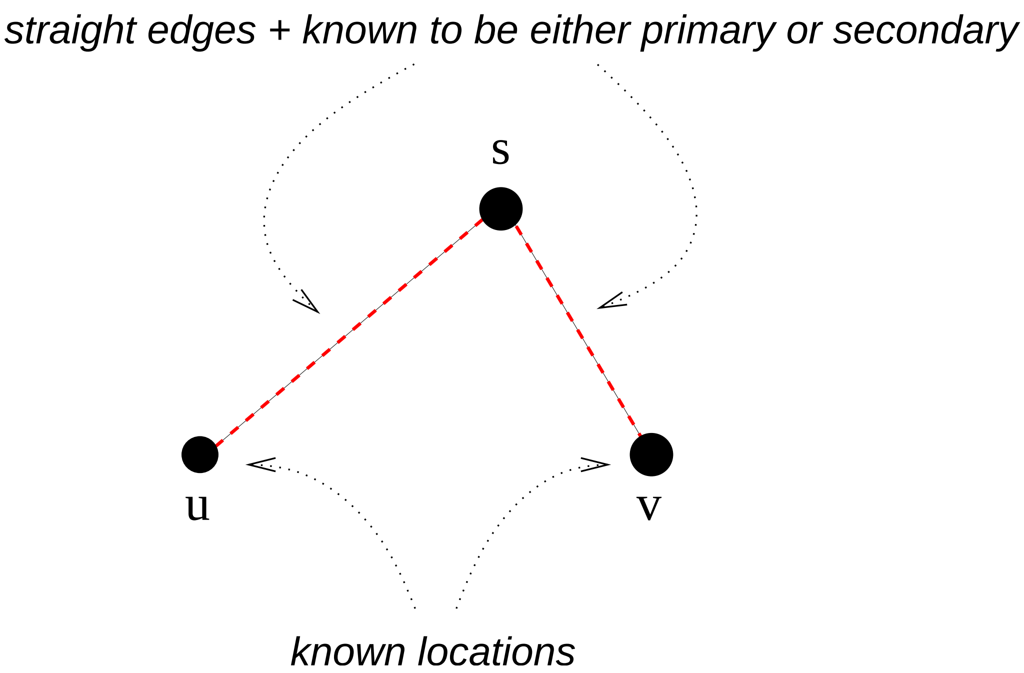
Merging Neighbouring Nodes

Lemma

Assume for a given topology $\mathcal{T}$ and direction set that

- the locations of two neighbours u and v of a Steiner point s are known
- edges (s, u) and (s, v) should be straight and have been labeled primary or secondary

Then, if s exists then its location is unique and can be computed in constant time.



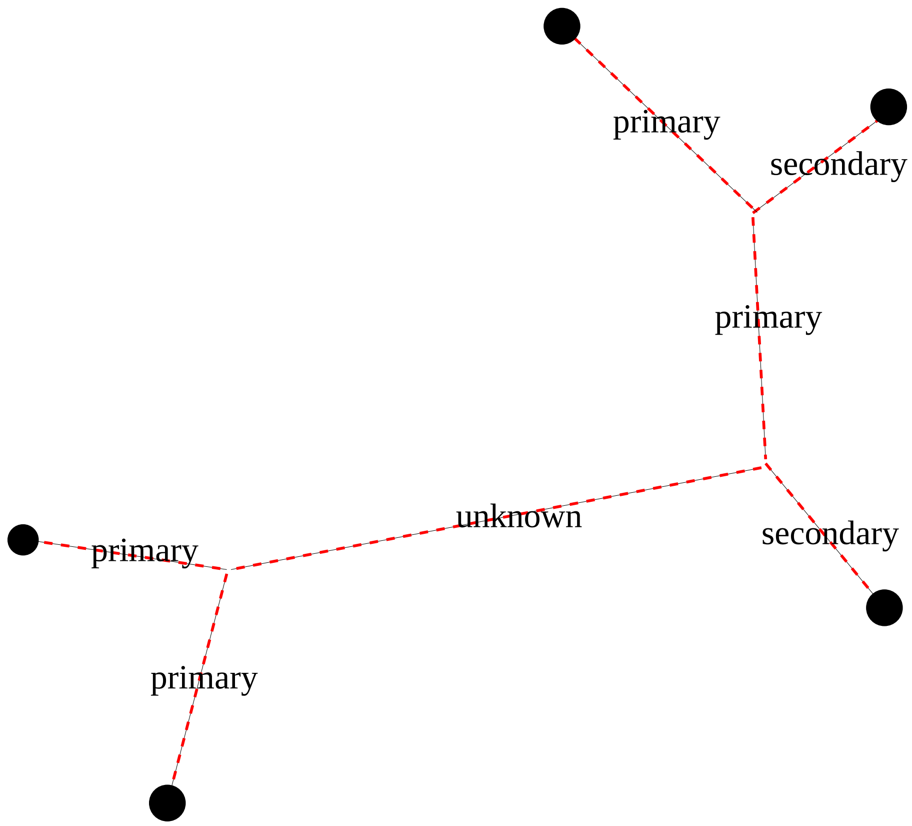
Construction of Steiner Tree with Known Labeling

Corollary

If a direction set is given and all edges except one in a topology $\mathcal{T}$ with n terminals have been labeled primary or secondary, then in $O(n)$ time we can either construct a full λ-SMT for $\mathcal{T}$ with the given labeling, or show that no such tree exists.

Proof

Root $\mathcal{T}$ at the unlabeled edge (=bent edge) and iteratively merge nodes bottom-up.



Quadratic Time Algorithm

Arbitrarily assign the numbers 1 to $2n - 3$ to the edges of $\mathcal{T}$. Consider some λ-SMT for $\mathcal{T}$. Move primary material to low-numbered edges using zero-shifts.

Now there exists a number k (with $1 \leq k \leq 2n - 3$) such that

- all edges numbered $1, \ldots, k - 1$ are primary
- all edges numbered $k + 1, \ldots, 2n - 3$ are secondary

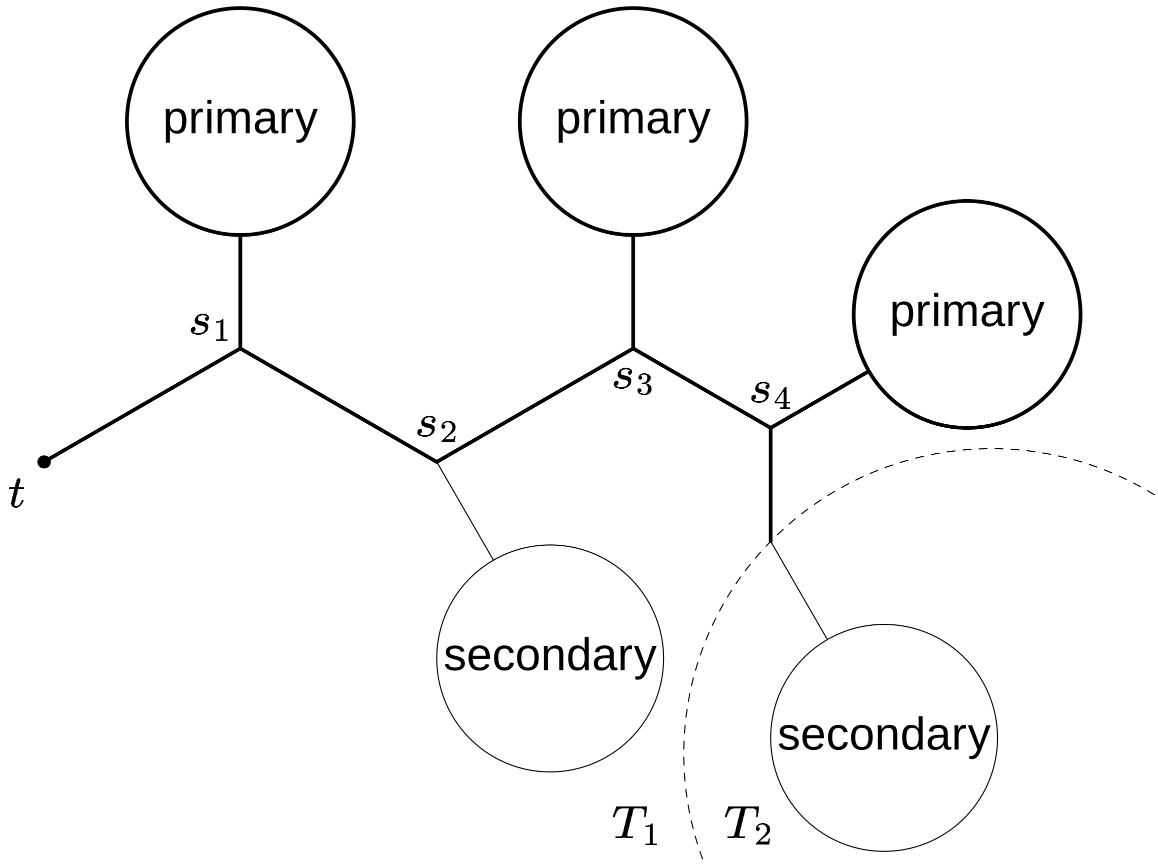
Gives a straightforward algorithm for computing a λ-SMT for $\mathcal{T}$: For each direction set and each choice of $k \in \{1, \ldots, 2n - 3\}$ use the $O(n)$ algorithm to compute a tree having the corresponding labeling.

Running time: $O(\lambda n^2)$

Canonical Forms

Any numbering of the edges gives a tree that is canonical with respect to the ordering.

Example: Choosing a root t and visiting the edges in a depth-first order gives a powerful canonical form:



Linear Time Algorithm

Assume that a direction set is given.

Initially we label all edges as being secondary, and define the locations of all Steiner points as undefined.

Root the topology $\mathcal{T}$ at some terminal t; let L[v] and R[v] denote the children of node v.

Perform two depth-first order traversals of $\mathcal{T}$ using the following algorithm:

TRAVERSE(u, v)
  if (2. traversal) **then** TRYBENTEDGE(u, v)
  if (v is a Steiner point) **then**
    TRAVERSE(v, L[v])
    TRAVERSE(v, R[v])
    $\Phi[v] = $ MERGERAYS(L[v], R[v])

Operation MERGERAYS(L[v], R[v]) attempts to compute the location of the parent based on the location of its children.

Linear Time Algorithm: Bent Edge Construction

TRYBENTEDGE(u, v)
  PS[u] = *primary*
  if ($u = r$) **then**
      Φ[u] = ray with source u having the same colour as Φ[v]
  else
      $x =$ P[u] and y = third neighbour of u
      Φ[u] = MERGERAYS(x,y)
  if (v is a terminal) **then**
      Φ[v] = ray with source v having the same colour as Φ[u]
  if (Φ[u] and Φ[v] have same colour and intersect at c) **then**
      c is corner point in constructed tree
  PS[v] = *primary*

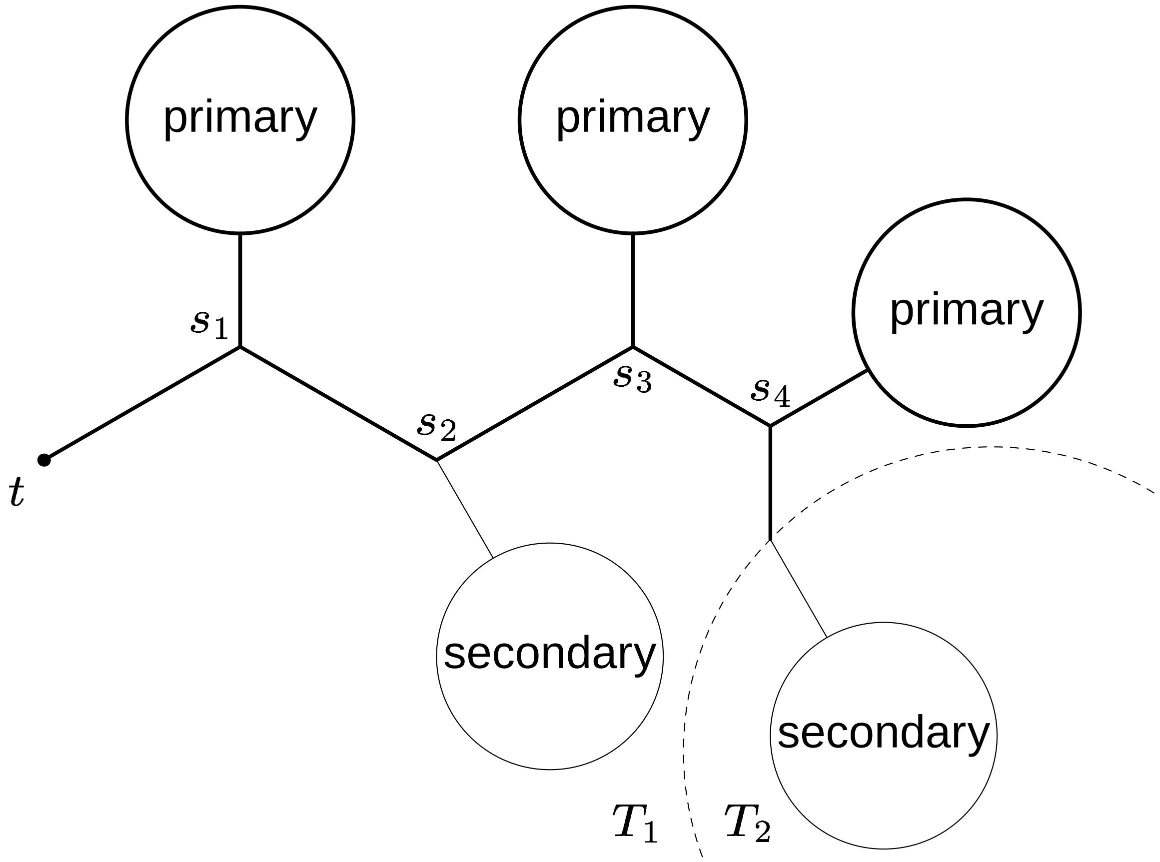
PS[u]: Labeling of edge from node u towards the (coming) bent edge

Φ[u]: Ray located at u with colour (=direction) of the edge from u towards the (coming) bent edge

Linear Time Algorithm: Correctness

Consider path P from the root t to the bent edge:

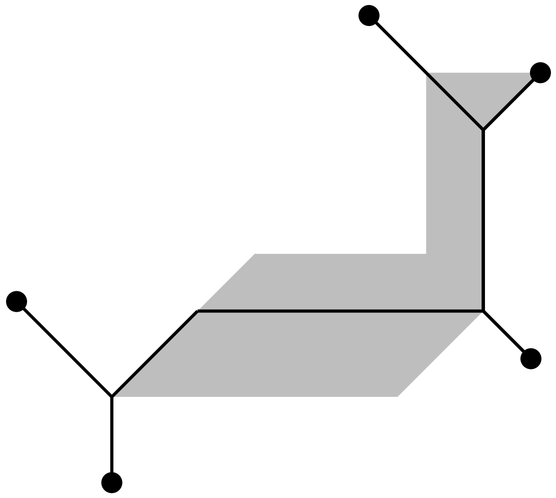
- all edges on P are primary
- all subtrees connected to P are either primary or secondary
- secondary subtrees (from 1. traversal) are merged with primary subtrees (from 2. traversal)



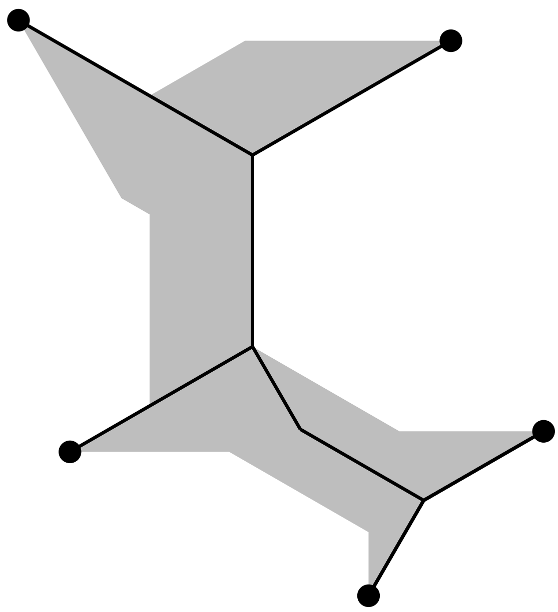
Flexibility Polygon

Observation: There usually exist (infinitely) many λ-SMTs for a given terminal set N and topology $\mathcal{T}$.

The union of all λ-SMTs is denoted the flexibility polygon for N and $\mathcal{T}$.



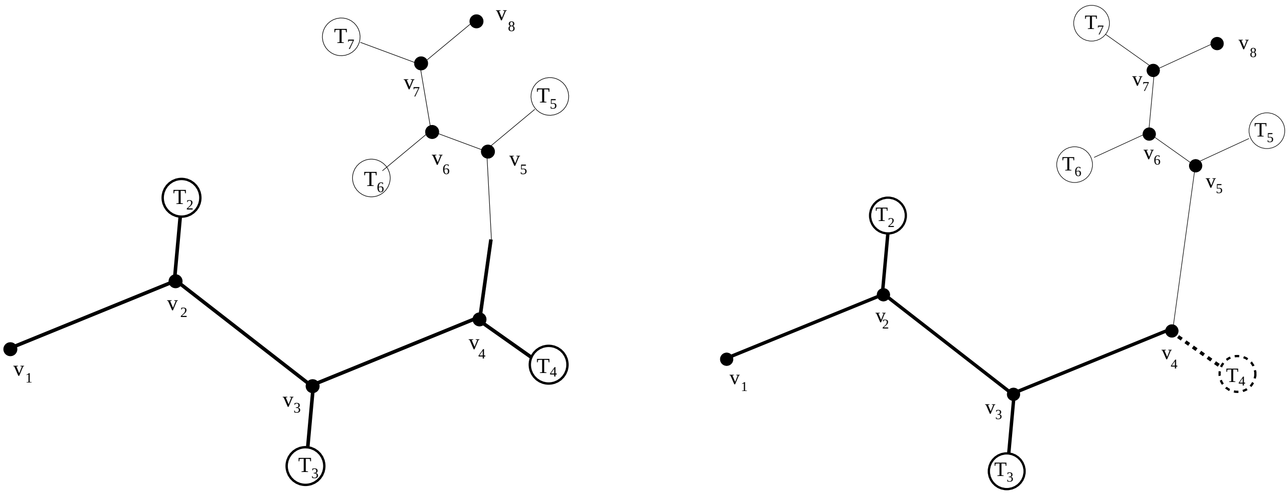
$\lambda = 4$



$\lambda = 6$

Rightmost paths

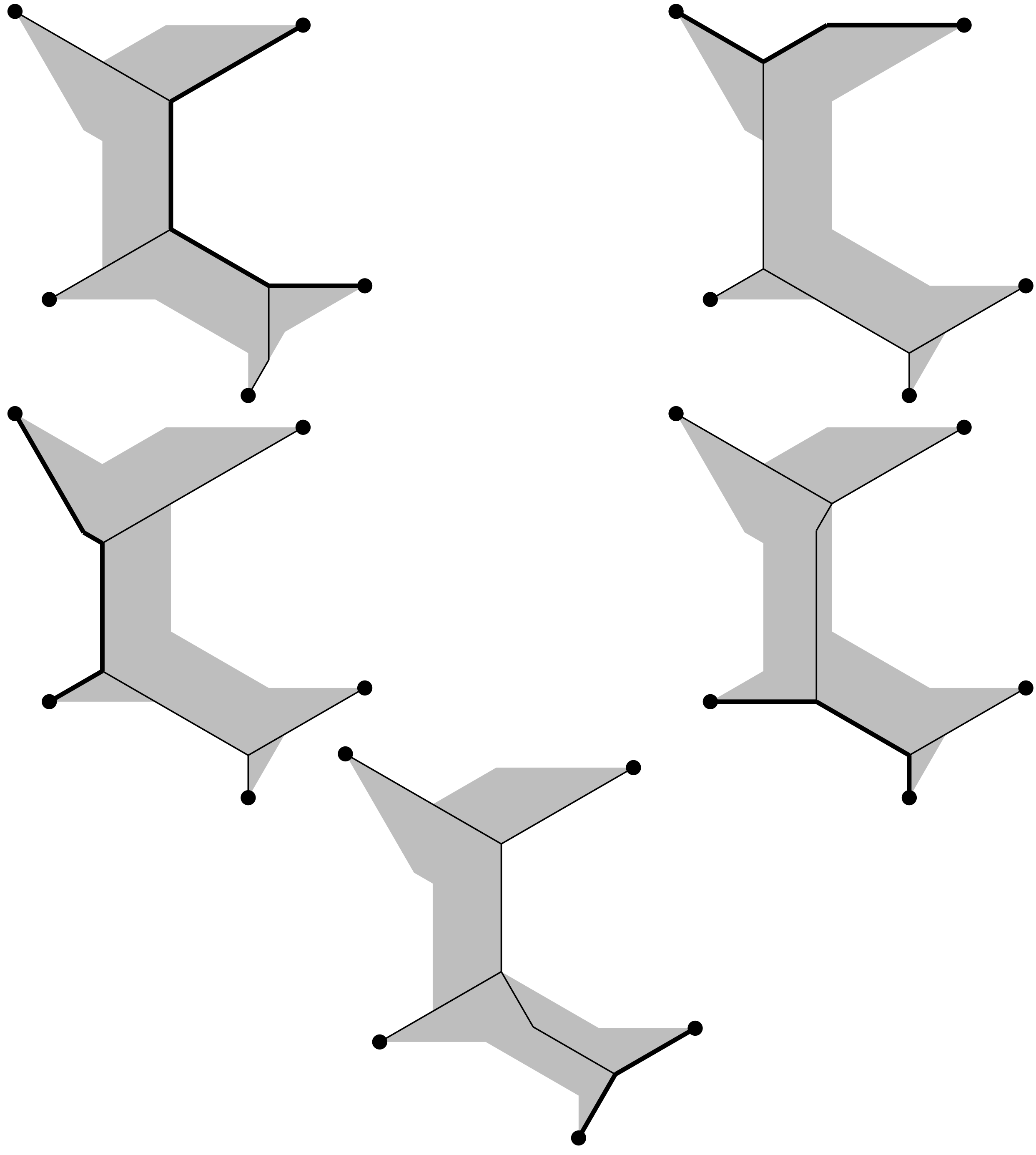
Consider a path P between two terminals in T. Push P as far as possible to the "right" (rightmost path). Resulting path P^r identical to path in canonical form tree resulting from depth-first ordering of edges in T with first terminal as root.



Theorem
For a given edge uv on P^r no embedding of the same edge in any other λ-SMT lies to the right of the oriented line through uv.

Lemma
Boundary of flexibility polygon consists of rightmost concave paths.

Boundary of Flexibility Polygon: Example



Construction of Flexibility Polygon

The following is a $O(\lambda n)$ time algorithm for constructing a flexibility polygon for N and $\mathcal{T}$.

Step 1 Construct a λ-SMT T for N and $\mathcal{T}$.

Use $O(\lambda n)$ algorithm just described.

Step 2 Construct all possible primary and secondary subtrees.

These form building blocks of rightmost concave paths.

Step 3 Construct boundary of flexibility polygon.

Form rightmost concave paths using building blocks.

Construction of Primary and Secondary Subtrees

CONSTRUCTSUBTREES(N, $\mathcal{T}$, T)

// Initialization phase
$Q = \emptyset$ // empty queue of oriented edges (=subtrees)
forall $[u, v] \in E(\mathcal{T})$ **do**
  if $u \in N$ **then**
    $\Phi_p[u, v] = u$; $p[u, v] = 0$
  else
    $\Phi_p[u, v] =$ NIL; $p[u, v] = \infty$
    Let v_1 and v_2 be the two neighbours of u other than v
    if $v_1 \in N$ and $v_2 \in N$ **then**
      ENQUEUE(Q, $[u, v]$) // u has two neighbouring terminals
// Construction phase
while $Q \neq \emptyset$
  $[u, v]$ = DEQUEUE(Q)
  Let v_1 and v_2 be the two neighbours of u other than v
  Let r be the intersection (if any) between
  the rays ($\Phi_p[v_1, u]$, $\Theta_p[v_1, u]$) and ($\Phi_p[v_2, u]$, $\Theta_p[v_2, u]$)
  if r exists **then**
    $p[u, v] = p[v_1, u] + p[v_2, u] + d^*(\Phi_p[v_1, u], r) + d^*(\Phi_p[v_2, u], r)$
    if $p(u, v) < p(T)$ **then**
      $\Phi_p[u, v] = r$ // subtree $\mathcal{T}[u, v]$ has now been constructed
      if v is a Steiner point **then**
        Let u_1 and u_2 be the two neighbours of v other than u
        if $\Phi_p[u_1, v] \neq$ NIL **then** ENQUEUE(Q, $[v, u_2]$)
        if $\Phi_p[u_2, v] \neq$ NIL **then** ENQUEUE(Q, $[v, u_1]$)

Flexibility Polygons for a Steiner Point

Union of all Steiner point locations for a given Steiner point in $\mathcal{T}$.