Articulated Figures

Paired Joint Coordinates, Denavit-Hartenberg Forward Kinematics, Key-Frames

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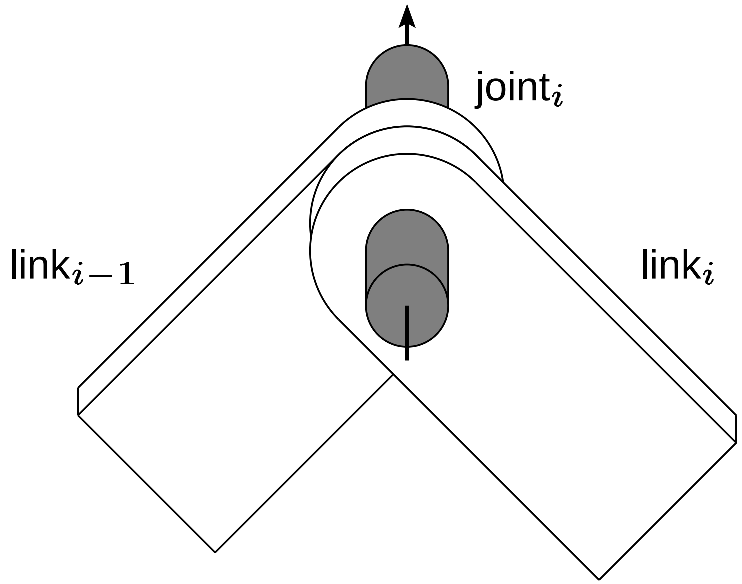
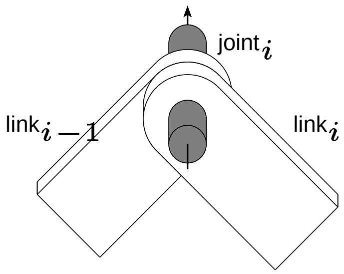
Articulated Figures



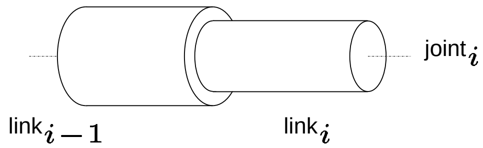
An *articulated figure* is a construction made of *links* and *joints*.

The different *links* are connected by *joints* which have some number of degrees of freedom.

Joint Types



(a) revolute joint

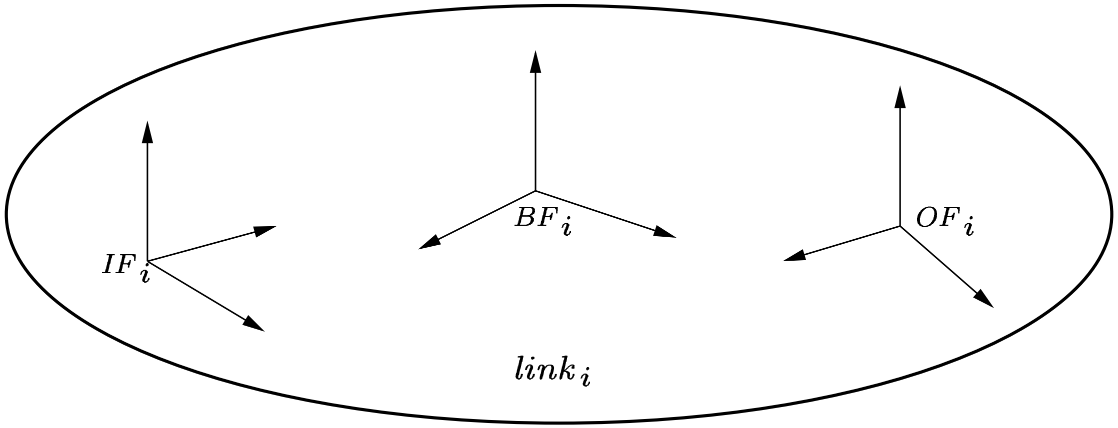


(b) prismatic joint

revolute joint: rotates around *one* axis.

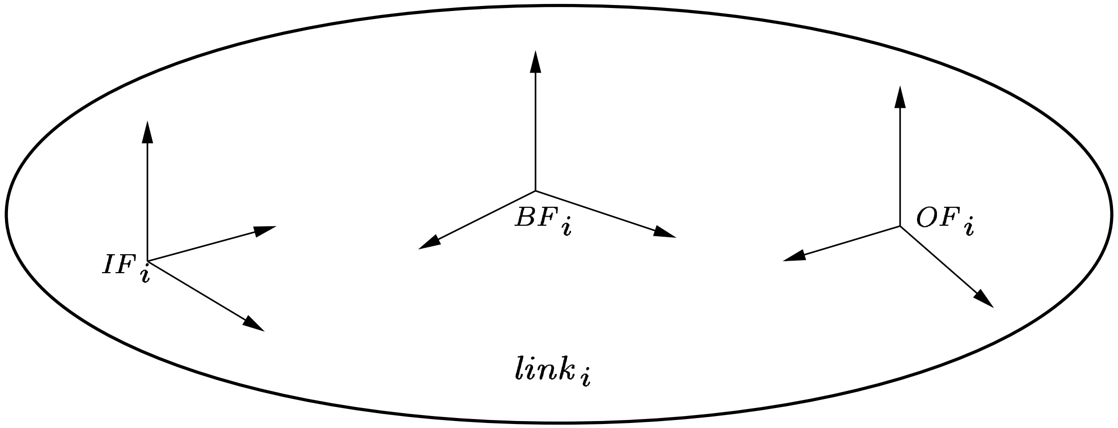
prismatic joint: translates along *one* axis.

Paired Joint Coordinates



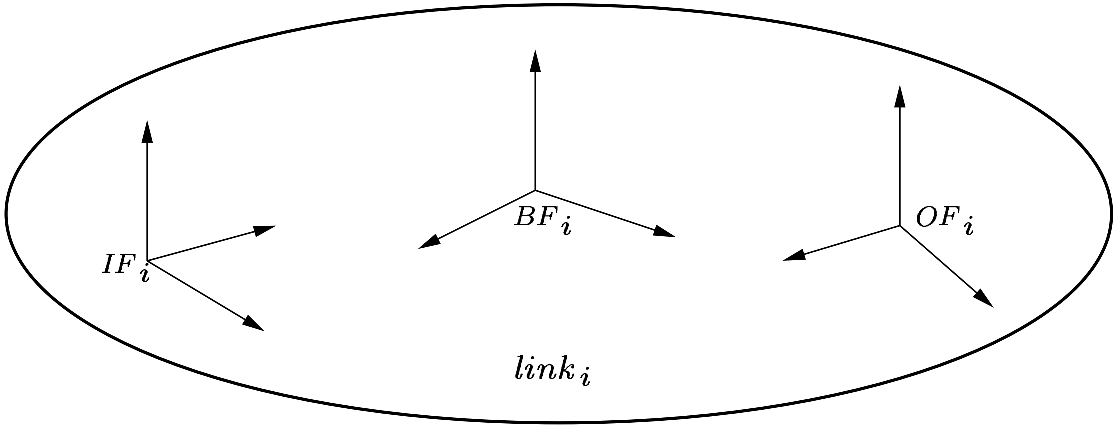
The Body Frame (BF_i)**:** *local* coordinate system associated with $link_i$. The geometry of $link_i$ is described in this local coordinate system. Generally, the origin of BF_i is chosen to be the *center of mass* of $link_i$.

Paired Joint Coordinates



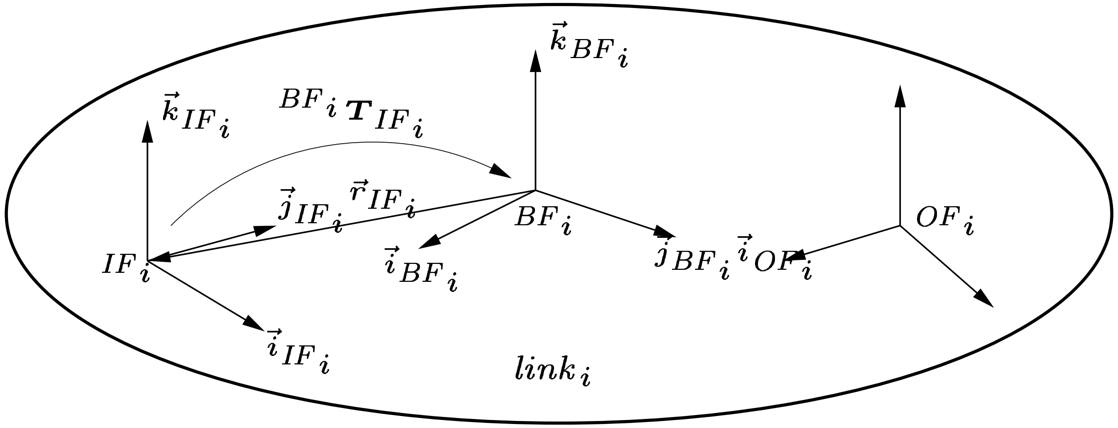
The Inner-Frame (IF_i): *local* coordinate system associated with $link_i$. Usually, its origin is located at $joint_i$ and has one axis parallel to the direction of motion of the joint. **Notice:** the origin and the axes are specified in the body frame (BF_i)

Paired Joint Coordinates



The Outer-Frame (OF_i)**:** *local* coordinate system associated with $link_i$. Usually, its origin is located at $joint_{i+1}$ and has one axis parallel to the direction of motion of the joint. **Notice:** the origin and the axes are specified in the body frame (BF_i)

The Transformation ${}^{BF_i}\boldsymbol{T}_{IF_i}$

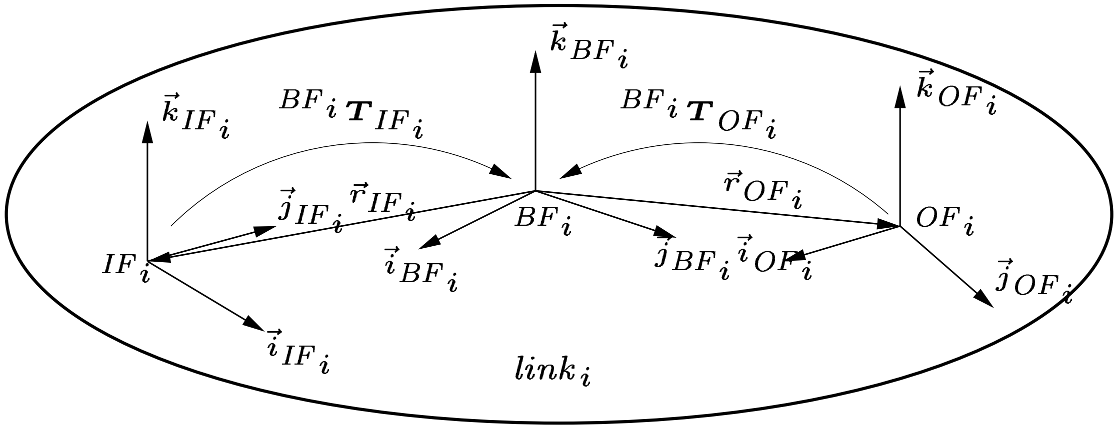


$${}^{BF_i}\boldsymbol{T}_{IF_i} = \boldsymbol{T}_{IF_i}(\vec{r}_{IF_i})\boldsymbol{R}_{IF_i}(\varphi, \vec{u})$$

$$\boldsymbol{T}_{IF_i}(\vec{r}_{IF_i}) = \begin{bmatrix} \boldsymbol{I} & \vec{r}_{IF_i} \\ \vec{0}^T & 1 \end{bmatrix}$$

$$\boldsymbol{R}_{IF_i}(\varphi, \vec{u}) = \begin{bmatrix} \vec{i}_{IF_i} & \vec{j}_{IF_i} & \vec{k}_{IF_i} & \vec{0} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Transformation ${}^{BF_i}\boldsymbol{T}_{OF_i}$

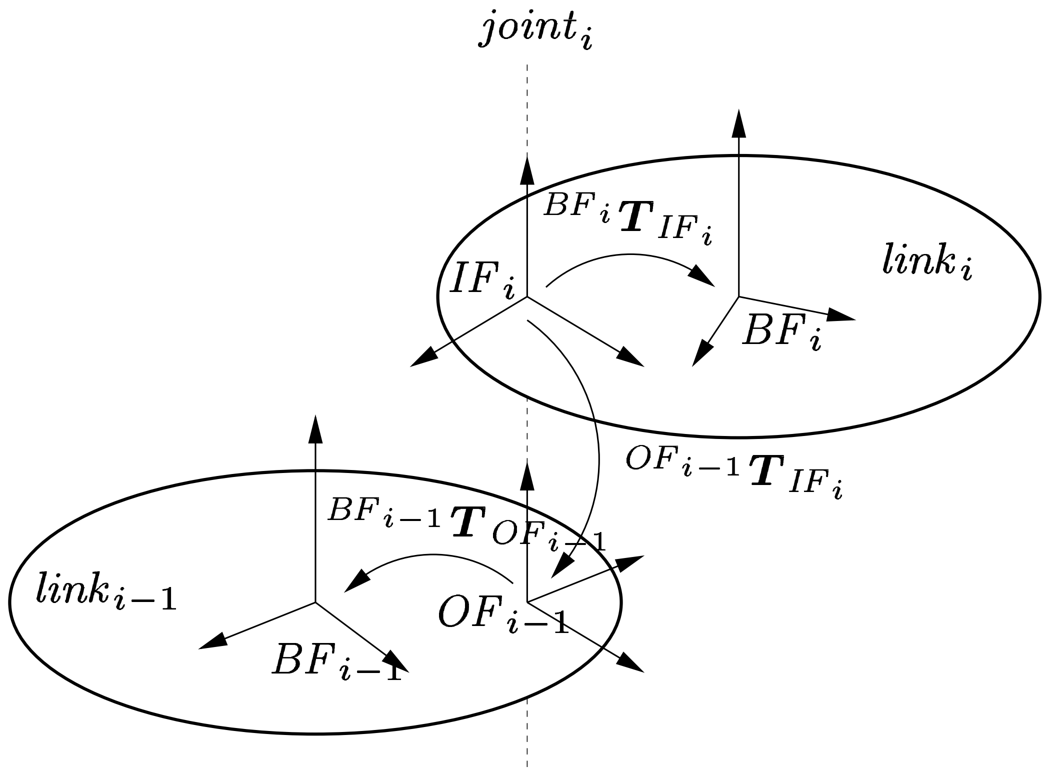


$$ {}^{BF_i}\boldsymbol{T}_{OF_i} = \boldsymbol{T}_{OF_i}(\vec{r}_{OF_i})\boldsymbol{R}_{OF_i}(\varphi, \vec{b}) $$

$$ \boldsymbol{T}_{OF_i}(\vec{r}_{OF_i}) = \begin{bmatrix} \boldsymbol{I} & \vec{r}_{OF_i} \\ \vec{0}^T & 1 \end{bmatrix} $$

$$ \boldsymbol{R}_{OF_i}(\varphi, \vec{c}) = \begin{bmatrix} \vec{i}_{OF_i} & \vec{j}_{OF_i} & \vec{k}_{OF_i} & \vec{0} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The Transformation ${}^{OF_{i-1}}\boldsymbol{T}_{IF_i}$



$$ {}^{OF_{i-1}}\boldsymbol{T}_{IF_i} = \boldsymbol{T}_i(\vec{d}_i)\boldsymbol{R}_i(\varphi_i, \vec{u}_i) $$

The Transformation ${}^{(i-1)}\boldsymbol{T}_i$ and ${}^{0}\boldsymbol{T}_N$

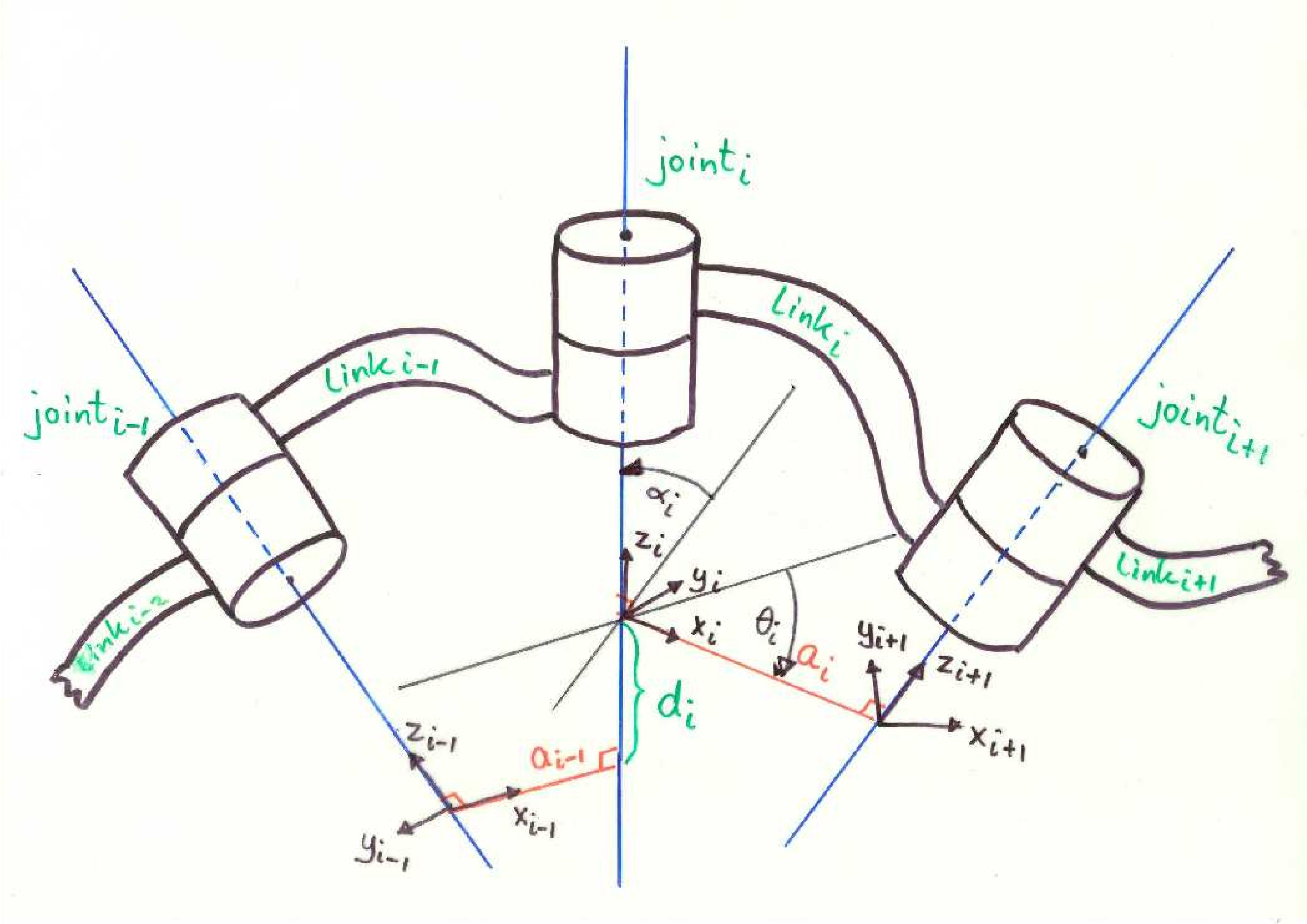
$$
\begin{aligned}
&{}^{(i-1)}\boldsymbol{T}_i(\vec{d}_i, \varphi_i, \vec{u}_i) = \\
&\qquad \left({}^{BF_{i-1}}\boldsymbol{T}_{OF_{i-1}}\right)\left({}^{OF_{i-1}}\boldsymbol{T}_{IF_i}(\vec{d}_i, \varphi_i, \vec{u}_i)\right)\left({}^{BF_i}\boldsymbol{T}_{IF_i}\right)^{-1}
\end{aligned}
$$

Generalized joint parameters $\vec{\theta}_i = (\vec{d}_i, \varphi_i, \vec{u}_i)$

$$
{}^{(i-1)}\boldsymbol{T}_i(\vec{\theta}_i) = \left({}^{BF_{i-1}}\boldsymbol{T}_{OF_{i-1}}\right)\left({}^{OF_{i-1}}\boldsymbol{T}_{IF_i}(\vec{\theta}_i)\right)\left({}^{BF_i}\boldsymbol{T}_{IF_i}\right)^{-1}
$$

$$
\begin{aligned}
{}^{0}\boldsymbol{T}_N &= {}^{0}\boldsymbol{T}_N(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\
&= {}^{0}\boldsymbol{T}_1(\vec{\theta}_1)\,{}^{1}\boldsymbol{T}_2(\vec{\theta}_2)\cdots{}^{(N-1)}\boldsymbol{T}_N(\vec{\theta}_N)
\end{aligned}
$$

Denavit-Hartenberg



Denavit-Hartenberg

link length	a_i	The perpendicular distance between axes of $joint_i$ and $joint_{i+1}$.

Denavit-Hartenberg

link twist	α_i	The angle between axes of $joint_i$ and $joint_{i+1}$. The angle α_i is measured around the $\vec{x}_i$ axis. Positive angles are measured counter clock wise when looking from the tip of vector $\vec{x}_i$ towards its foot.

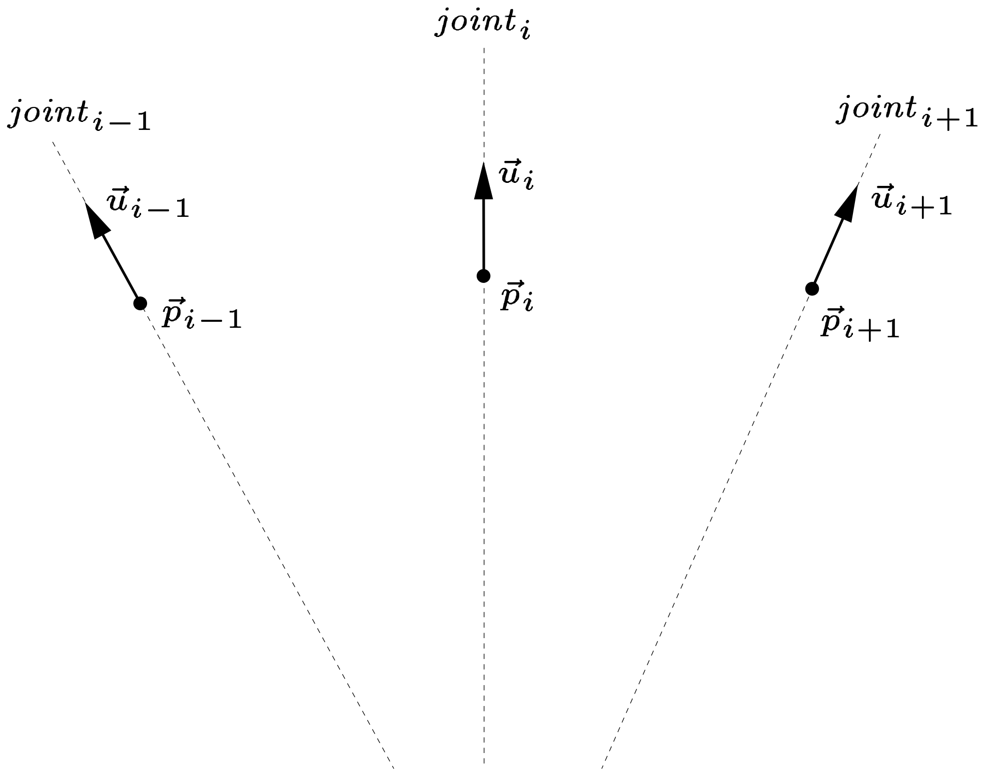
Denavit-Hartenberg

link offset	d_i	The distance between the origins of the coordinate frames attached to joint $joint_{i-1}$ and $joint_i$ measured along the axis of $joint_i$. For a prismatic joint this is a joint parameter.

Denavit-Hartenberg

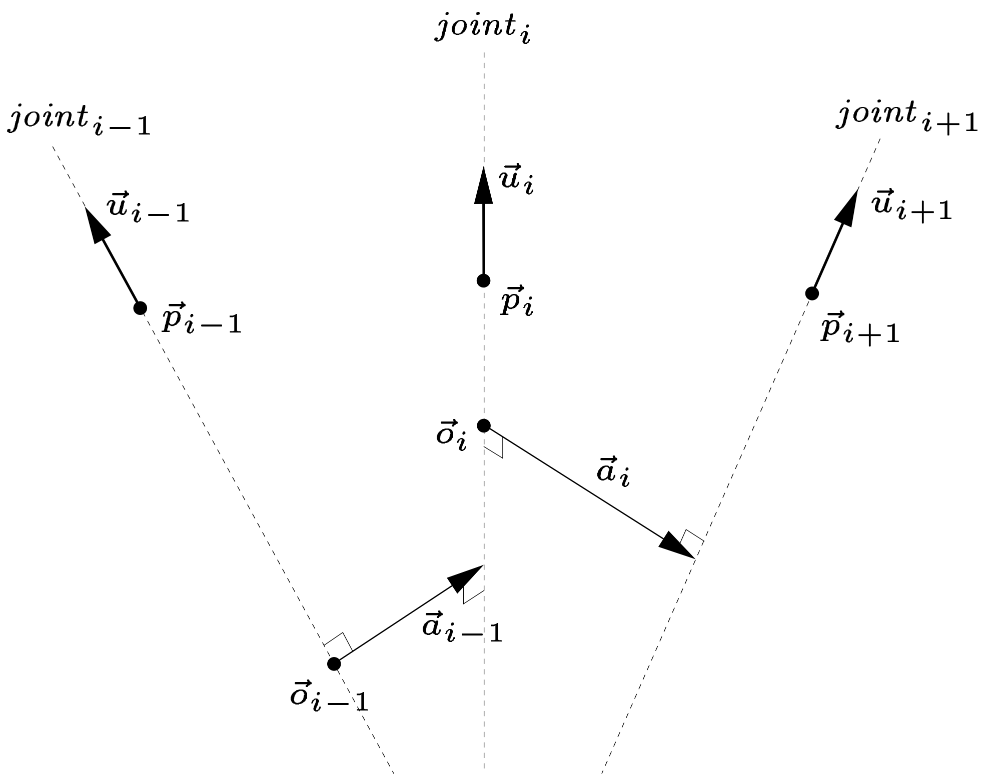
joint angle	φ_i	The angle between the link lengths $\vec{a}_{i-1}$ and $\vec{a}_i$. The angle φ_i is measured around the $\vec{z}_i$ axis. Positive angles are measured counter clock wise when looking from the tip of vector $\vec{z}_i$ towards its foot. For a revolute joint this is a joint parameter.

Joint Axes



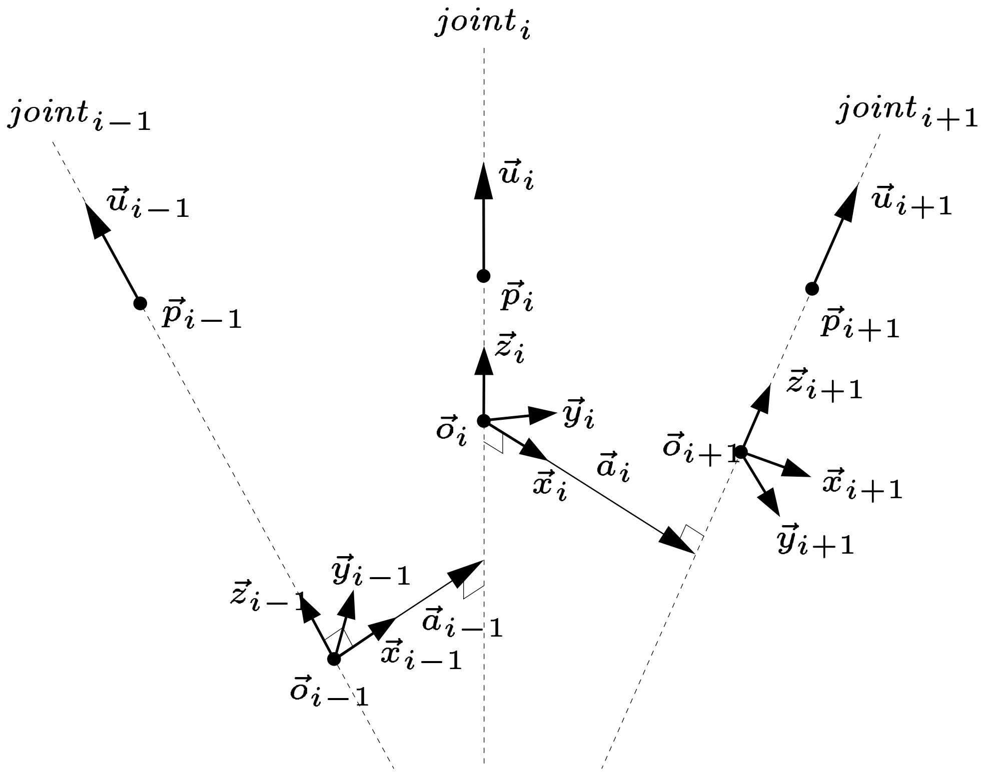
Joint axes $\vec{l}_i(s) = \vec{p}_i + s\vec{u}_i$.

Common Perpendiculars



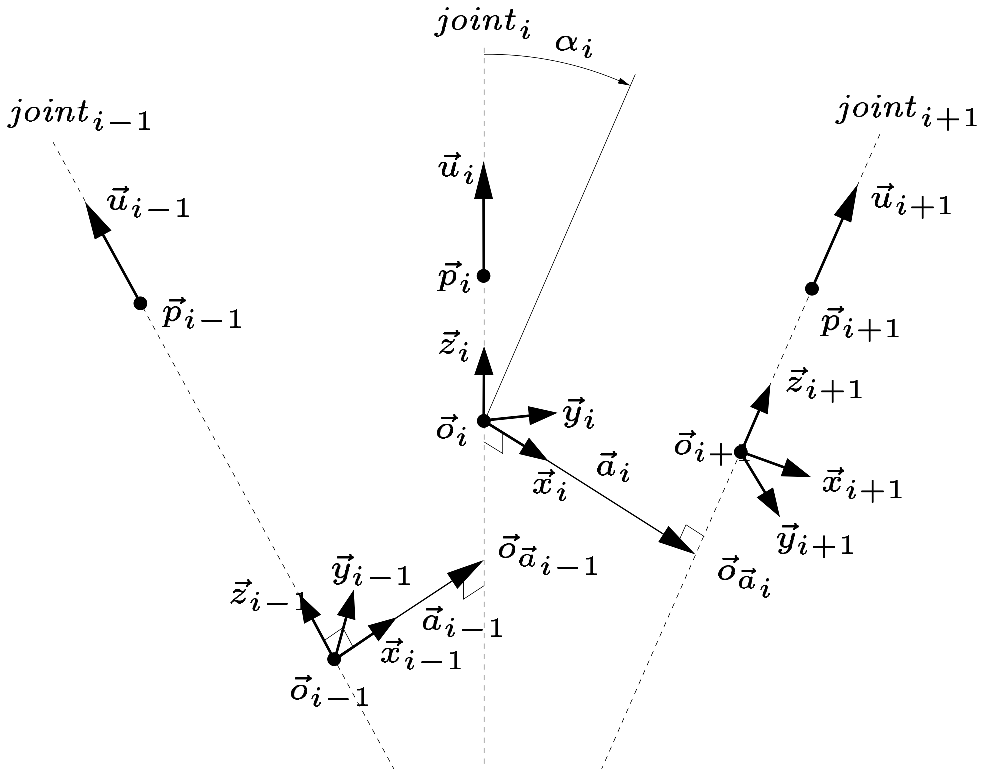
Link vectors $\vec{a}_i$ and the origins $\vec{o}_i$.

Coordinate-Frame Attachment



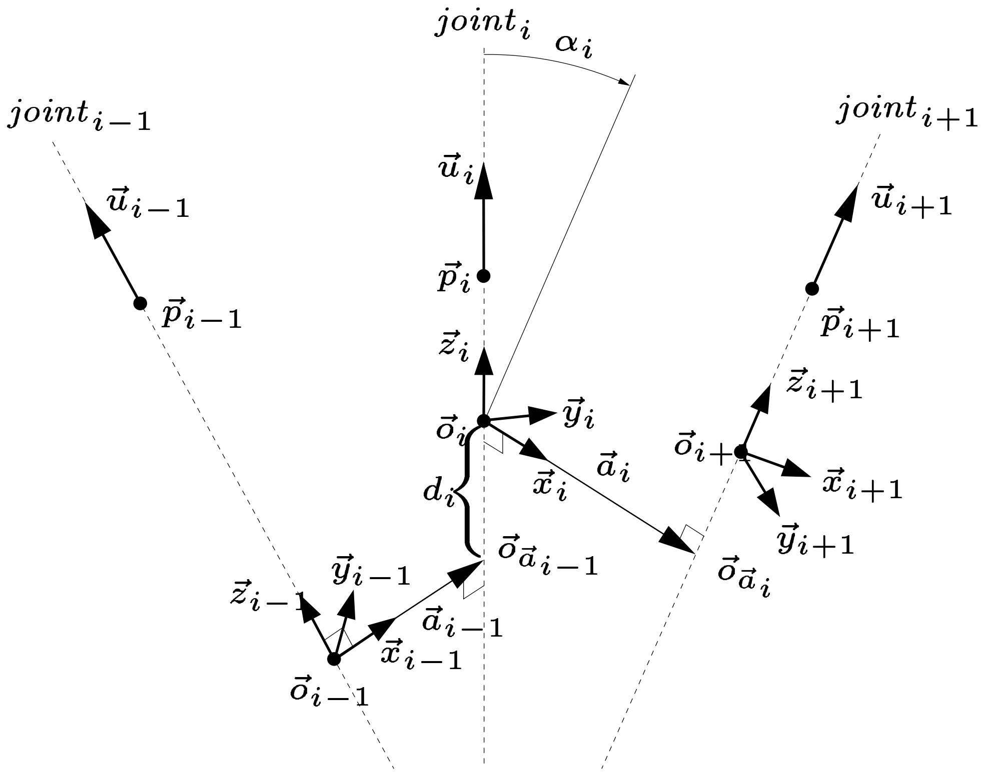
Link-frames $\vec{o}_i, \vec{x}_i, \vec{y}_i, \vec{z}_i$.

Link Twist



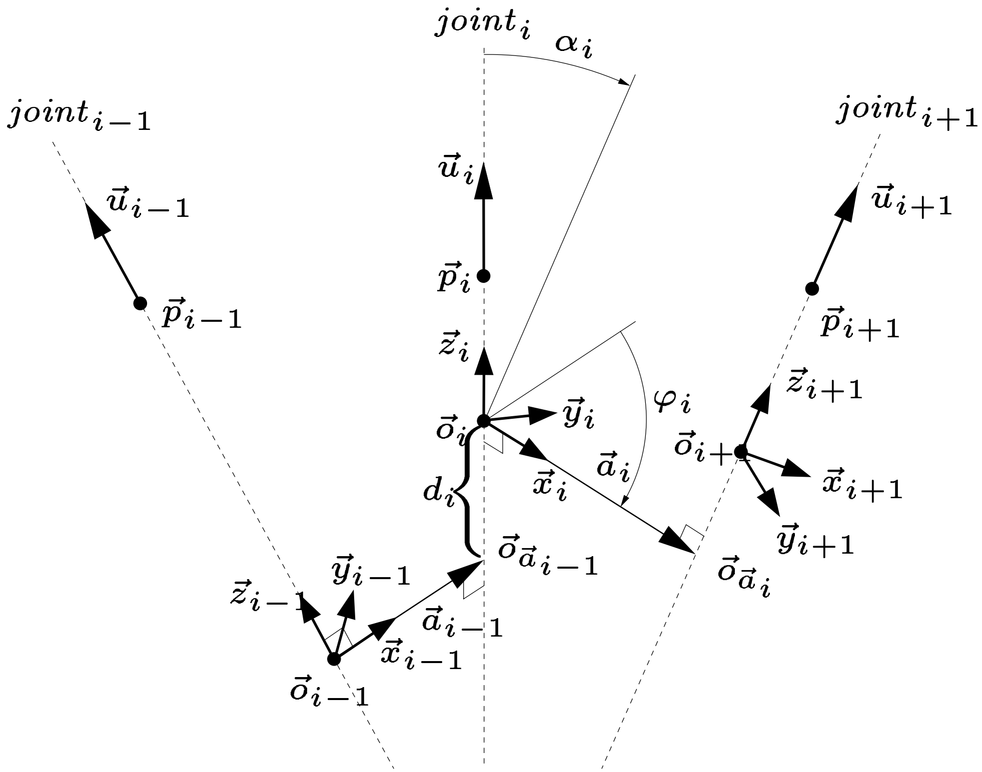
Link frames and link twist α_i. The link twist α_i is negative.

Link Offset



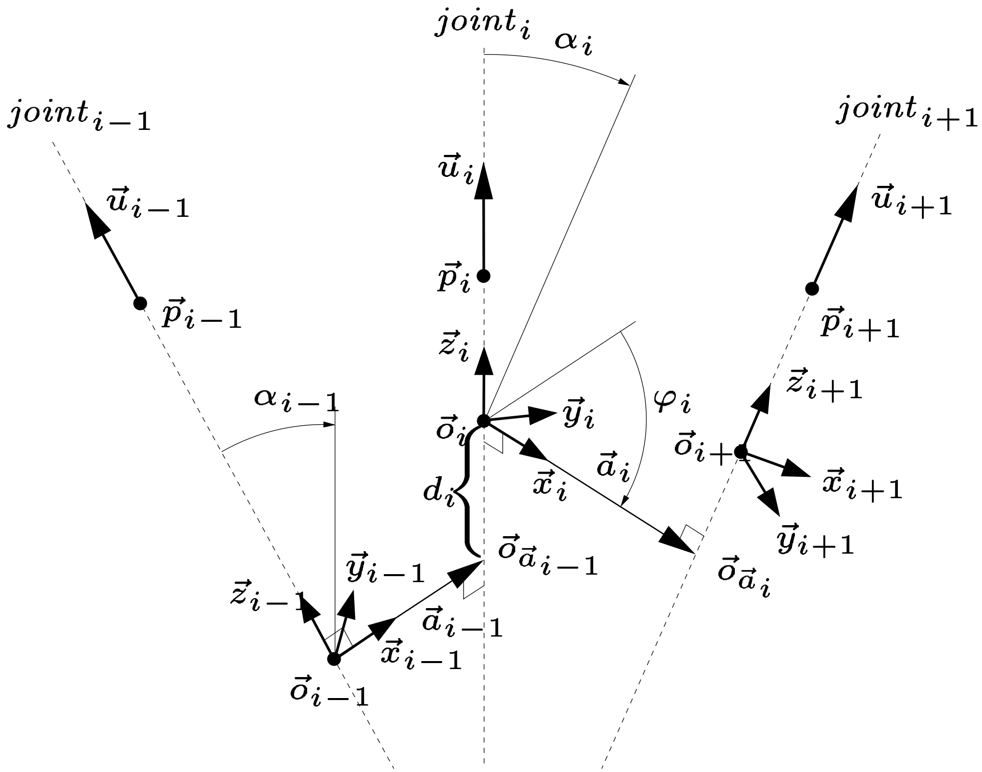
The link offset is the distance between the origins of joint $joint_{i-1}$ and $joint_i$.

Joint Angle



The coordinate frames of the links and the joint angle. In the figure, the joint angle φ_i is negative.

The Transformation ${}^{(i-1)}T_i$



The coordinate frames of the links and the joint angle.

The Transformation ${}^{(i-1)}\boldsymbol{T}_i$

1. Rotate the angle φ_i around the axis $\vec{z}_i$.
2. Translate the distance d_i along the axis $\vec{z}_i$.
3. Translate the distance a_{i-1} along the axis $\vec{x}_i$.
4. Rotate the angle α_{i-1} around the axis $\vec{x}_i$.

$$
\begin{aligned}
{}^{(i-1)}\boldsymbol{T}_i(\varphi_i, d_i, a_{i-1}, \alpha_{i-1}) = \\
\boldsymbol{R}_{\vec{x}_i}(\alpha_{i-1})\boldsymbol{T}_{\vec{x}_i}(a_{i-1})\boldsymbol{T}_{\vec{z}_i}(d_i)\boldsymbol{R}_{\vec{z}_i}(\varphi_i)
\end{aligned}
$$

The Transformation ${}^{(i-1)}\boldsymbol{T}_i$

$$
{}^{(i-1)}\boldsymbol{T}_i(d_i, \varphi_i) =
$$

$$
\begin{bmatrix}
\cos\varphi_i & -\sin\varphi_i & 0 & a_{i-1} \\
\cos\alpha_{i-1}\sin\varphi_i & \cos\alpha_{i-1}\cos\varphi_i & -\sin\alpha_{i-1} & -d_i\sin\alpha_{i-1} \\
\sin\alpha_{i-1}\sin\varphi_i & \sin\alpha_{i-1}\cos\varphi_i & \cos\alpha_{i-1} & d_i\cos\alpha_{i-1} \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

The Transformation ${}^{(i-1)}\boldsymbol{T}_i$

$$
\begin{aligned}
{}^0\boldsymbol{T}_N &= {}^0\boldsymbol{T}_N(d_1, \varphi_1, \ldots, d_N, \varphi_N) \\
&= {}^0\boldsymbol{T}_1(d_1, \varphi_1)\,{}^1\boldsymbol{T}_2(d_2, \varphi_2) \cdots {}^{N-1}\boldsymbol{T}_N(d_N, \varphi_N)
\end{aligned}
$$

generalized joint-parameters $\vec{\theta}_i = (d_i, \varphi_i)$.

$$
\begin{aligned}
{}^0\boldsymbol{T}_N &= {}^0\boldsymbol{T}_N(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\
&= {}^0\boldsymbol{T}_1(\vec{\theta}_1)\,{}^1\boldsymbol{T}_2(\vec{\theta}_2) \cdots {}^{(N-1)}\boldsymbol{T}_N(\vec{\theta}_N)
\end{aligned}
$$

The End Effector

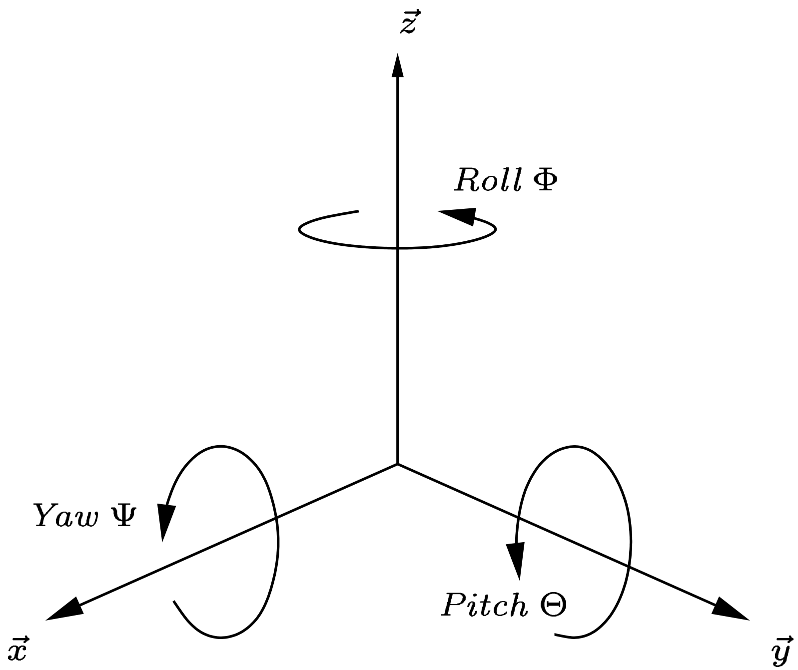
$$
\begin{aligned}
{}^{0}\boldsymbol{T}_N &= {}^{0}\boldsymbol{T}_N(\vec{\theta}_1, \ldots, \vec{\theta}_N) = {}^{0}\boldsymbol{T}_1(\vec{\theta}_1)\,{}^{1}\boldsymbol{T}_2(\vec{\theta}_2)\cdots{}^{(N-1)}\boldsymbol{T}_N(\vec{\theta}_N) \\
&= \begin{bmatrix}
m_{11}(\vec{\theta}_1, \ldots, \vec{\theta}_N) & m_{12}(\vec{\theta}_1, \ldots, \vec{\theta}_N) & m_{13}(\vec{\theta}_1, \ldots, \vec{\theta}_N) & m_{14}(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\
m_{21}(\vec{\theta}_1, \ldots, \vec{\theta}_N) & m_{22}(\vec{\theta}_1, \ldots, \vec{\theta}_N) & m_{23}(\vec{\theta}_1, \ldots, \vec{\theta}_N) & m_{24}(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\
m_{31}(\vec{\theta}_1, \ldots, \vec{\theta}_N) & m_{32}(\vec{\theta}_1, \ldots, \vec{\theta}_N) & m_{33}(\vec{\theta}_1, \ldots, \vec{\theta}_N) & m_{34}(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{0}\boldsymbol{T}_N &= {}^{0}\boldsymbol{T}_N(\vec{\theta}_1, \ldots, \vec{\theta}_N) = {}^{0}\boldsymbol{T}_1(\vec{\theta}_1)\,{}^{1}\boldsymbol{T}_2(\vec{\theta}_2)\cdots{}^{(N-1)}\boldsymbol{T}_N(\vec{\theta}_N) \\
&= \boldsymbol{T}(\vec{p})\boldsymbol{R}_z(\Phi)\boldsymbol{R}_y(\Theta)\boldsymbol{R}_x(\Psi)
\end{aligned}
$$

Location of the End Effector

$$\boldsymbol{T}(\vec{p}) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orientation of the End Effector



Any orientation can be specified by three rotations, the *Roll*, *Pitch*, and *Yaw* around the $\vec{x}, \vec{y}$ and $\vec{z}$ axes respectively.

Orientation of the End Effector

1. Yaw Ψ Rotate the angle Ψ around the $\vec{x}$ axis.
2. Pitch Θ Rotate the angle Θ around the $\vec{y}$ axis.
3. Roll Φ Rotate the angle Φ around the $\vec{z}$ axis.

Yaw Transformation $\boldsymbol{Y}(\Psi) = \boldsymbol{R}_{\vec{x}}(\Psi)$

$$\boldsymbol{Y}(\Psi) = \boldsymbol{R}_x(\Psi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Psi & -\sin\Psi & 0 \\ 0 & \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pitch transformation $P(\Theta) = R_{\vec{y}}(\Theta)$

$$\boldsymbol{Y}(\Psi) = \boldsymbol{R}_x(\Psi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Psi & -\sin\Psi & 0 \\ 0 & \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Roll transformation $\boldsymbol{R}(\Phi) = \boldsymbol{R}_{\vec{z}}(\Phi)$

$$\boldsymbol{R}(\Phi) = \boldsymbol{R}_z(\Phi) = \begin{bmatrix} \cos\Phi & -\sin\Phi & 0 & 0 \\ \sin\Phi & \cos\Phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Roll, Pitch, Yaw Transformation

$$\boldsymbol{T}_{RPY}(\Phi, \Theta, \Psi) = \boldsymbol{R}_z(\Phi)\boldsymbol{R}_y(\Theta)\boldsymbol{R}_x(\Psi)$$

$$= \begin{bmatrix} c\Phi & -s\Phi & 0 & 0 \\ s\Phi & c\Phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\Theta & 0 & s\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -s\Theta & 0 & c\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\Psi & -s\Psi & 0 \\ 0 & s\Psi & c\Psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\Phi c\Theta & c\Phi s\Theta s\Psi - s\Phi c\Psi & c\Phi s\Theta c\Psi + s\Phi s\Psi & 0 \\ s\Phi c\Theta & s\Phi s\Theta s\Psi + c\Phi c\Psi & s\Phi s\Theta c\Psi - c\Phi s\Psi & 0 \\ -s\Theta & c\Theta s\Psi & c\Theta c\Psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The End Effector ${}^0\boldsymbol{T}_N(\vec{p}, \Phi, \Theta, \Psi)$

$$
\begin{aligned}
{}^0\boldsymbol{T}_N &= {}^0\boldsymbol{T}_N(\vec{p}, \Phi, \Theta, \Psi) = \boldsymbol{T}(\vec{p})\boldsymbol{T}_{RPY}(\Phi, \Theta, \Psi) \\
&= \boldsymbol{T}(\vec{p})\boldsymbol{R}(\Phi)\boldsymbol{P}(\Theta)\boldsymbol{Y}(\Psi) = \boldsymbol{T}(\vec{p})\boldsymbol{R}_z(\Phi)\boldsymbol{R}_y(\Theta)\boldsymbol{R}_x(\Psi) \\
&= \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} c\Phi & -s\Phi & 0 & 0 \\ s\Phi & c\Phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} c\Theta & 0 & s\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -s\Theta & 0 & c\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\Psi & -s\Psi & 0 \\ 0 & s\Psi & c\Psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\Phi c\Theta & c\Phi s\Theta s\Psi - s\Phi c\Psi & c\Phi s\Theta c\Psi + s\Phi s\Psi & p_x \\ s\Phi c\Theta & s\Phi s\Theta s\Psi + c\Phi c\Psi & s\Phi s\Theta c\Psi - c\Phi s\Psi & p_y \\ -s\Theta & c\Theta s\Psi & c\Theta c\Psi & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Computation of $\vec{p}, \Phi, \Theta, \Psi$

$$
{}^0\boldsymbol{T}_N = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\Phi c\Theta & c\Phi s\Theta s\Psi - s\Phi c\Psi & c\Phi s\Theta c\Psi + s\Phi s\Psi & p_x \\ s\Phi c\Theta & s\Phi s\Theta s\Psi + c\Phi c\Psi & s\Phi s\Theta c\Psi - c\Phi s\Psi & p_y \\ -s\Theta & c\Theta s\Psi & c\Theta c\Psi & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Computation of $\vec{p}$

$$\vec{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} m_{14} \\ m_{24} \\ m_{34} \end{bmatrix}$$

Computation of Ψ

$$\Psi = \arctan\left(\frac{m_{32}}{m_{33}}\right)$$

That is because the following relations hold

$$\begin{aligned}\Psi &= \arctan\left(\frac{m_{32}}{m_{33}}\right)\\ &= \arctan\left(\frac{\cos\Theta\sin\Psi}{\cos\Theta\cos\Psi}\right) = \arctan\left(\frac{\sin\Psi}{\cos\Psi}\right)\end{aligned}$$

Computation of Θ

$$\Theta = \arctan\left(\frac{-m_{31}}{m_{32}\sin\Psi + m_{33}\cos\Psi}\right)$$

This can be seen because the following relations hold

$$\begin{aligned}
\Theta &= \arctan\left(\frac{-m_{31}}{m_{32}\sin\Psi + m_{33}\cos\Psi}\right) \\
&= \arctan\left(\frac{\sin\Theta}{\cos\Theta\sin^2\Psi + \cos\Theta\cos^2\Psi}\right) \\
&= \arctan\left(\frac{\sin\Theta}{\cos\Theta(\sin^2\Psi + \cos^2\Psi)}\right) \qquad = \arctan\left(\frac{\sin\Theta}{\cos\Theta}\right)
\end{aligned}$$

Computation of Φ

$$\Phi = \arctan\left(\frac{m_{21}}{m_{11}}\right)$$

That is because the following relations hold

$$\Phi = \arctan\left(\frac{m_{21}}{m_{11}}\right) = \arctan\left(\frac{\sin\Phi\cos\Theta}{\cos\Phi\cos\Theta}\right)$$
$$= \arctan\left(\frac{\sin\Phi}{\cos\Phi}\right)$$

The State Vector

$$\vec{s}(\vec{\theta}_1, \ldots, \vec{\theta}_N) = \begin{bmatrix} p_x(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\ p_y(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\ p_z(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\ \Psi(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\ \Theta(\vec{\theta}_1, \ldots, \vec{\theta}_N) \\ \Phi(\vec{\theta}_1, \ldots, \vec{\theta}_N) \end{bmatrix}$$

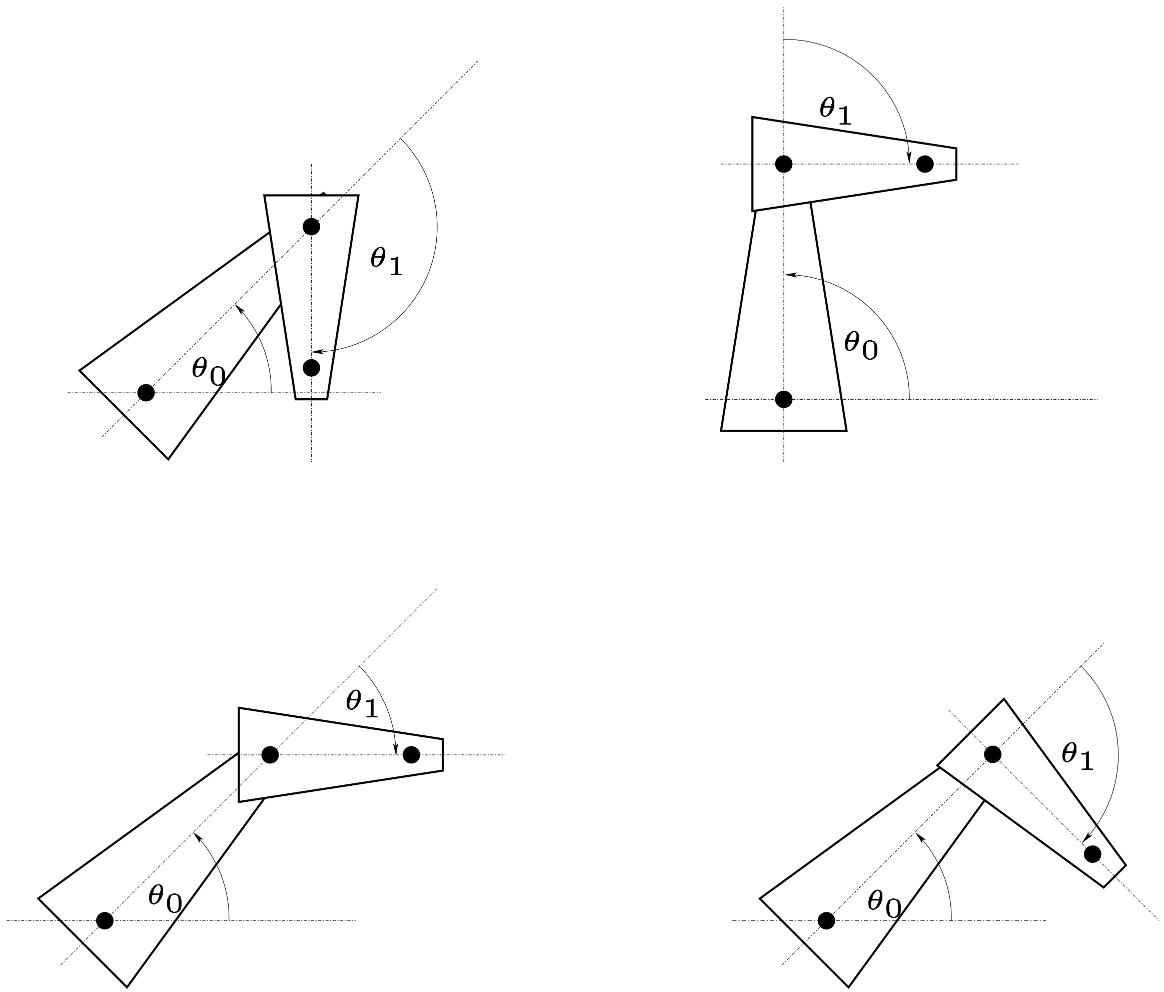
Direct Forward Kinematics

$$^{0}\boldsymbol{T}_N(\vec{\theta}_1,\ldots,\vec{\theta}_N) = {}^{0}\boldsymbol{T}_1(\vec{\theta}_1)\cdots{}^{(i-1)}\boldsymbol{T}_i(\vec{\theta}_i)\cdots{}^{(N-1)}\boldsymbol{T}_N(\vec{\theta}_N) =$$

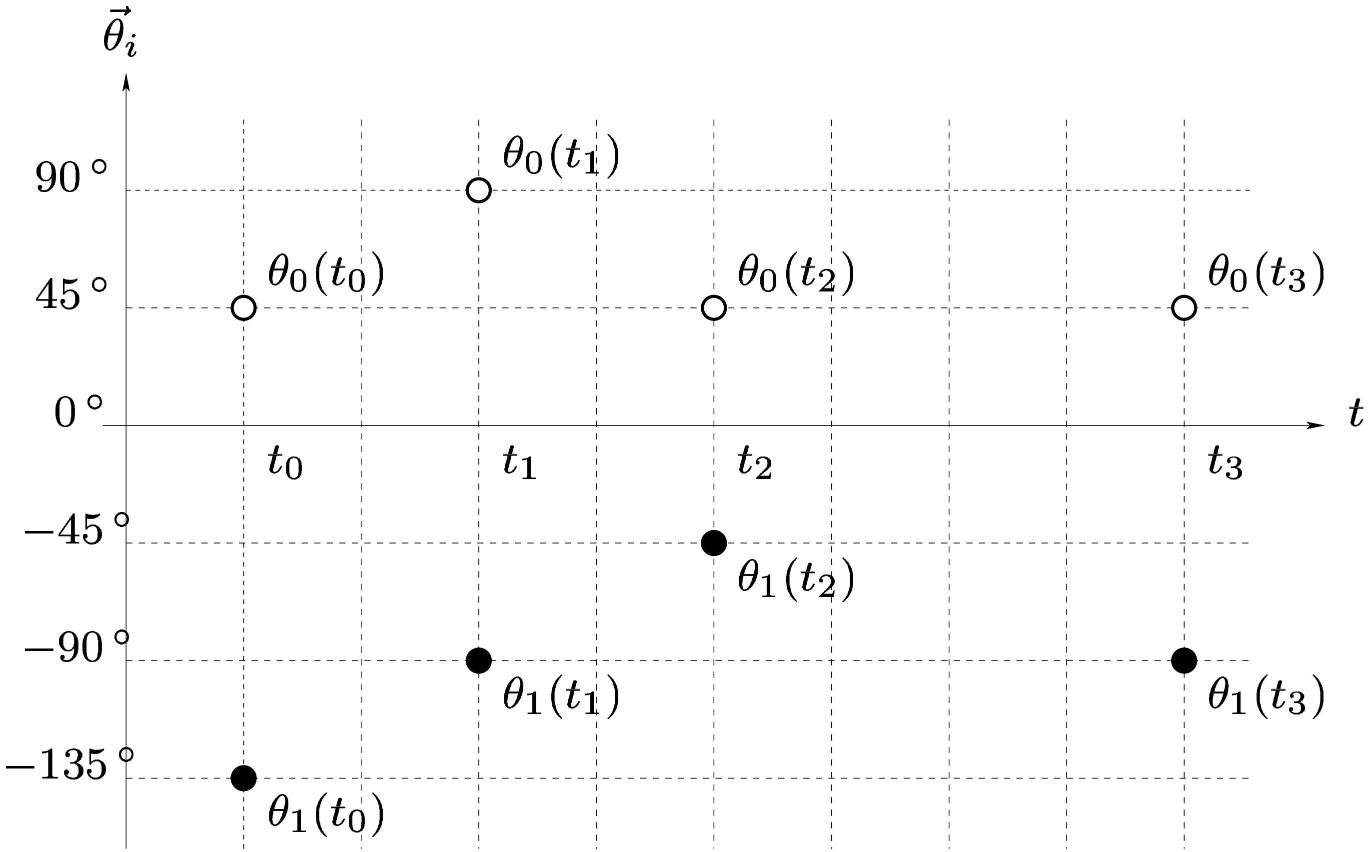
$$\begin{bmatrix} m_{11}(\vec{\theta}_1,\ldots,\vec{\theta}_N) & m_{12}(\vec{\theta}_1,\ldots,\vec{\theta}_N) & m_{13}(\vec{\theta}_1,\ldots,\vec{\theta}_N) & m_{14}(\vec{\theta}_1,\ldots,\vec{\theta}_N) \\ m_{21}(\vec{\theta}_1,\ldots,\vec{\theta}_N) & m_{22}(\vec{\theta}_1,\ldots,\vec{\theta}_N) & m_{23}(\vec{\theta}_1,\ldots,\vec{\theta}_N) & m_{24}(\vec{\theta}_1,\ldots,\vec{\theta}_N) \\ m_{31}(\vec{\theta}_1,\ldots,\vec{\theta}_N) & m_{32}(\vec{\theta}_1,\ldots,\vec{\theta}_N) & m_{33}(\vec{\theta}_1,\ldots,\vec{\theta}_N) & m_{34}(\vec{\theta}_1,\ldots,\vec{\theta}_N) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{s}(\vec{\theta}_1,\ldots,\vec{\theta}_N) = \begin{bmatrix} p_x(\vec{\theta}_1,\ldots,\vec{\theta}_N) \\ p_y(\vec{\theta}_1,\ldots,\vec{\theta}_N) \\ p_z(\vec{\theta}_1,\ldots,\vec{\theta}_N) \\ \Psi(\vec{\theta}_1,\ldots,\vec{\theta}_N) \\ \Theta(\vec{\theta}_1,\ldots,\vec{\theta}_N) \\ \Phi(\vec{\theta}_1,\ldots,\vec{\theta}_N) \end{bmatrix}$$

Key-Framing

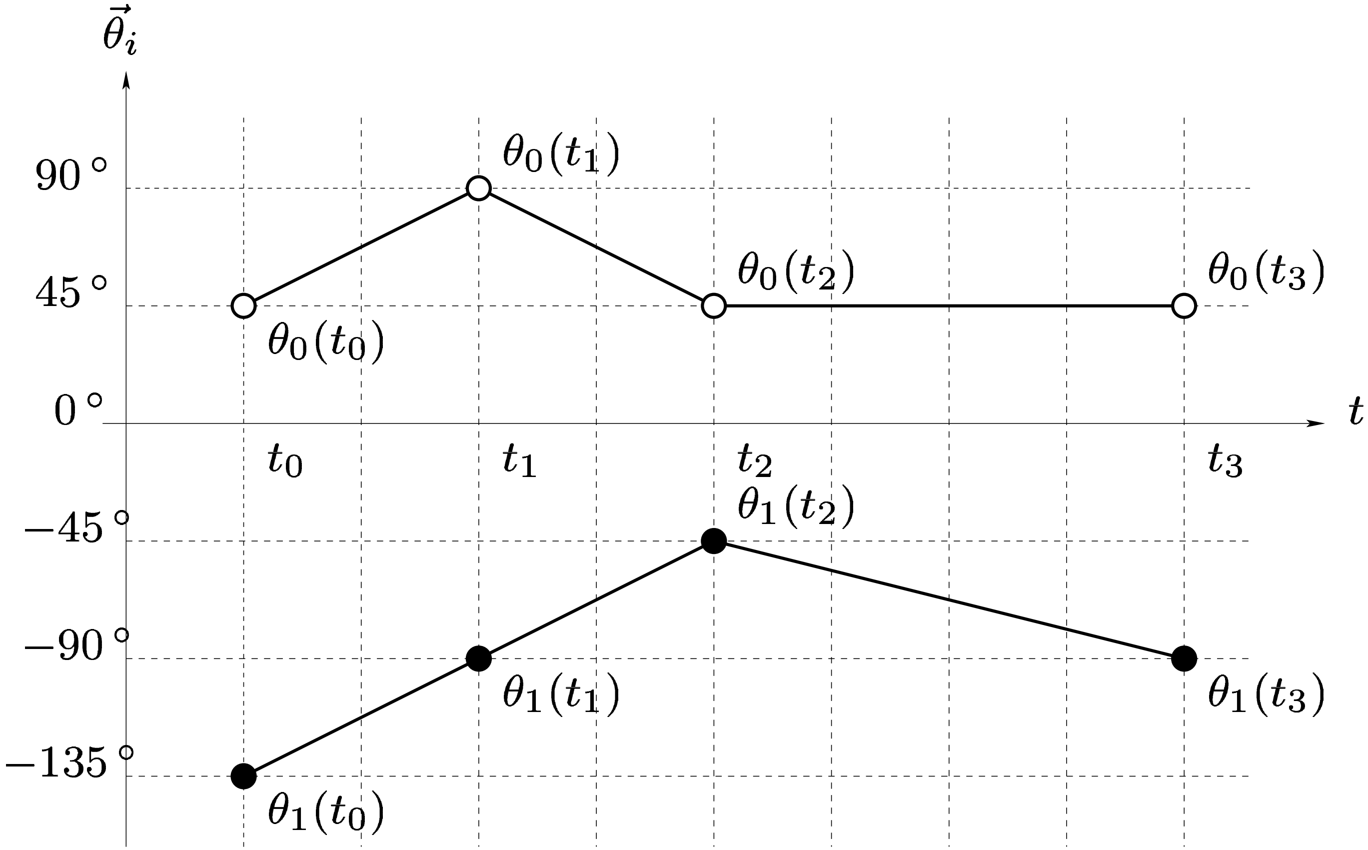


Key-Framing



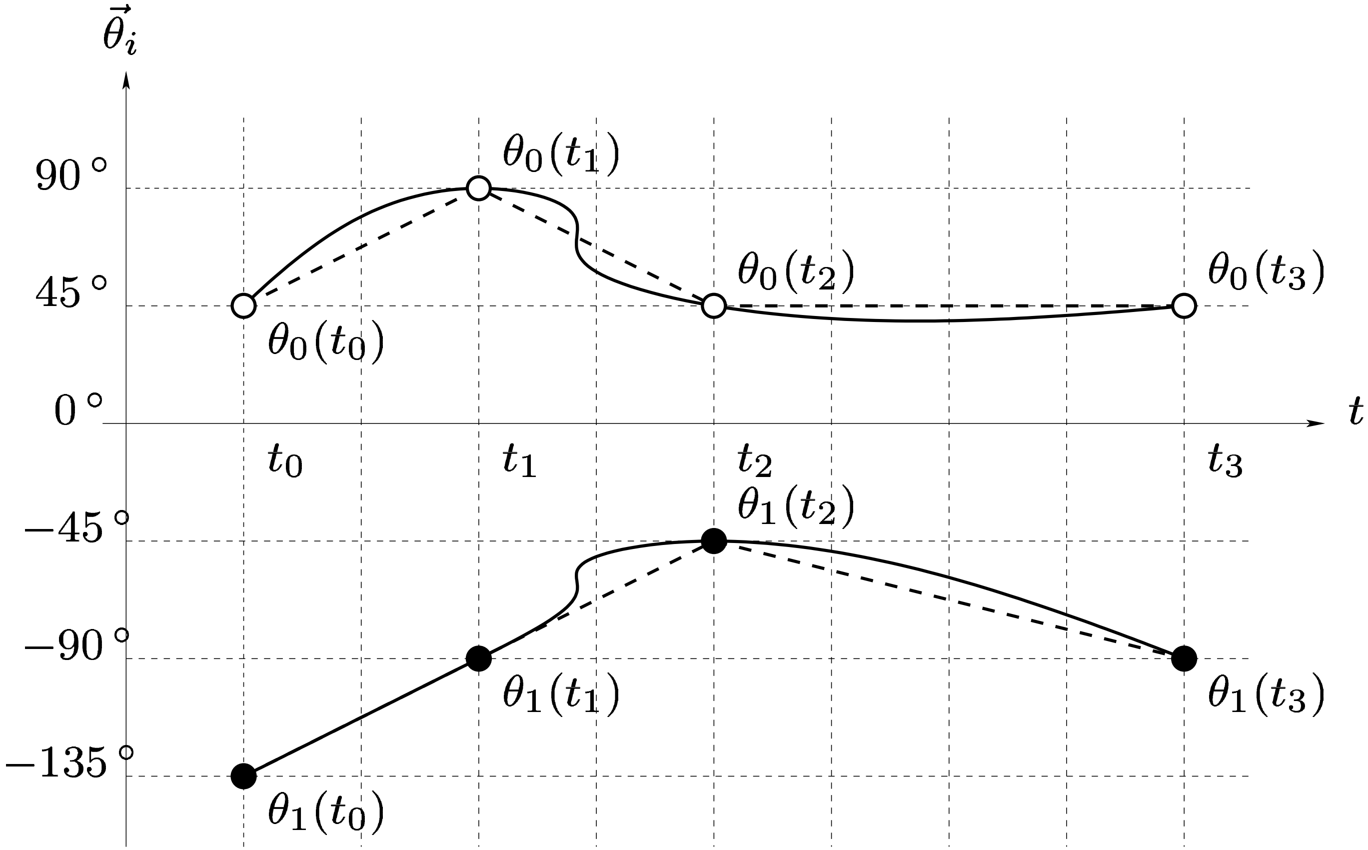
The joint parameters of the key-frames plotted against time.

Key-Framing



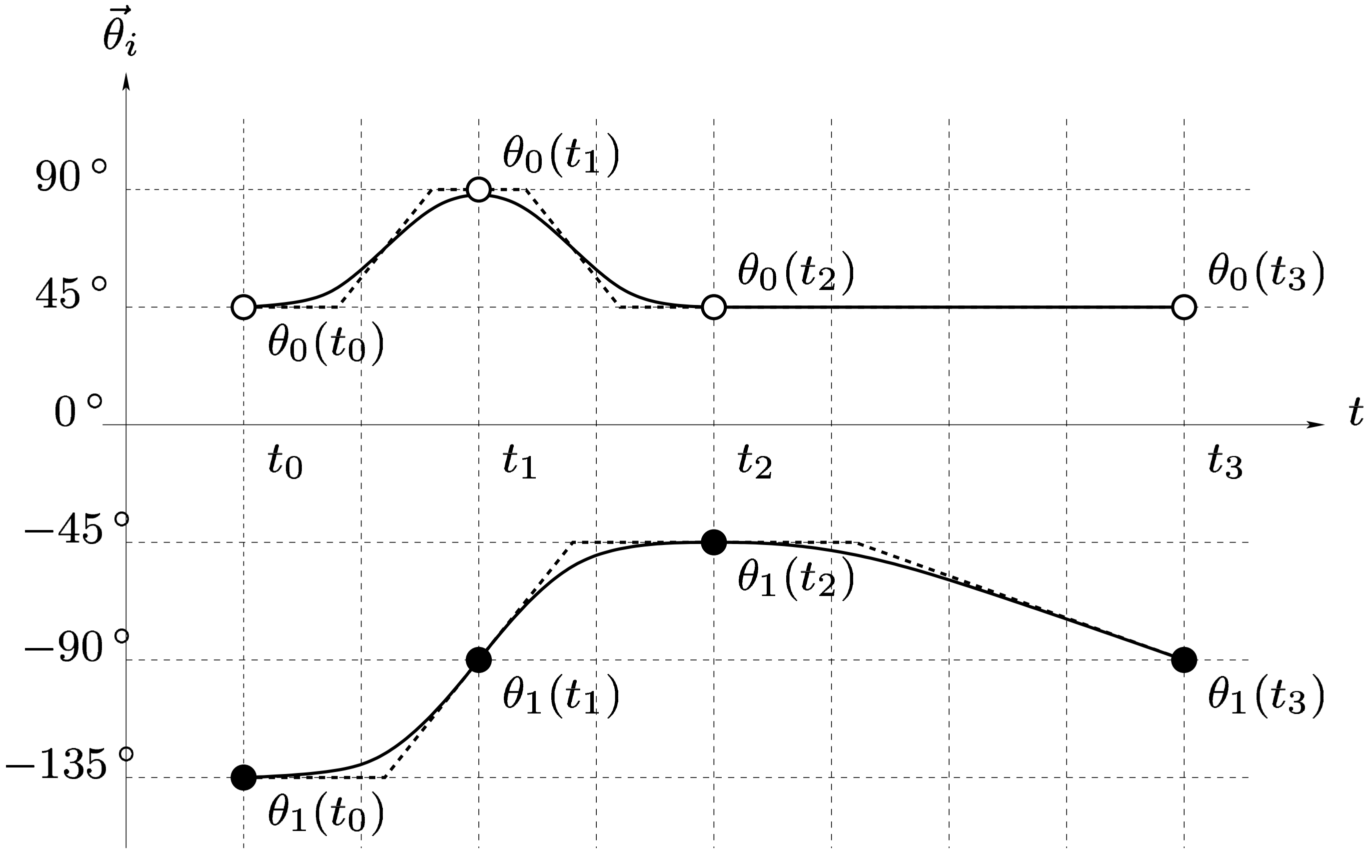
The joint parameters are interpolated linearly between the key-frames.

Key-Framing



The joint parameters are interpolated between the key-frames by a smooth interpolating spline.

Key-Framing



Interpolating spline with extra control points.