Written exam, 19 January 2005

Solution (Q1-Q10)

Linear Programming (Q1-Q9)

Initial slack form: \[ z = 0 + 3x_1 + \alpha x_2 \]
\[ x_3 = 4 - x_1 \]
\[ x_4 = 12 - (1/\beta)x_2 \]
\[ x_5 = 6 \Delta - 3x_1 - 2x_2 \]

Q1: \[ \gamma_1 = 3 - 1 - 3 = -1 \]
\[ \gamma_2 = \alpha - (1/\beta) - 2 \]
\[ \Rightarrow \quad 1B \]

Q2: The feasible region is bounded by the rectangle \((0,0), (4,0), (4,12\beta), (0,12\beta)\).

\[ C_s \text{ redundant: } (4,12\beta) \text{ is the only vertex with } x_1, x_2 > 0. \text{ The line representing } C_s \text{ will otherwise intersect the rectangle either at } (0,0) \text{ or at two distinct points of which some will have both } x_1 \text{ and } x_2 > 0. \]
\[ \Rightarrow \quad 2B \]

Q3: One possibility is to check each of the 10 subsets separately and realize that the answer is "no" for \((x_1, x_2, x_3) = (x_1, x_2, x_3) \text{ or } (x_2, x_3, x_4). \text{ A clever shortcut, however, is to look at the slack form which tells us that } x_4 = x_2 = 0 \text{ cannot occur and likewise for } x_3 \text{ and } x_4. \]
\[ \Rightarrow \quad 3B \]
Q4: z must equal $3x_1 + \alpha x_2$. We obtain:

$$6\Delta + 12\beta(\alpha - 2) = 3(2\Delta - 8\beta) + \alpha x_2 \quad \text{or} \quad 12\beta\alpha = \alpha x_2 \quad \text{or} \quad x_2 = 12\beta$$

$\Rightarrow \ 4C$

Q5: No nonbasic variable must have a positive coefficient in the objective function (the z-equation) $\Rightarrow \alpha \geq 2$.

No basic variable must be negative.

$$x_1: \ 2\Delta - 8\beta \geq 0 \quad \text{or} \quad \Delta \geq 4\beta$$

$$x_5: \ 4 - (2\Delta - 8\beta) \geq 0 \quad \text{or} \quad \Delta \leq 2 + 4\beta$$

$$\left\{ \begin{array}{l}
0 \leq \Delta - 4\beta \leq 2
\end{array} \right.$$  

$\Rightarrow \ 5C$

Q6: The feasible region is bounded, cf. Q2. Regardless of the values of $\beta, \gamma$, $P(\alpha, \beta, \Lambda)$ itself cannot be "unbounded" and $D(\alpha, \beta, \Lambda)$ can accordingly not be "infeasible". Note that the value of $\alpha$ is of no significance in this context.

$\Rightarrow \ 6E$

Q7: The lines representing the three constraints intersect at three points called $S_{12}$, $S_{13}$, $S_{23}$, respectively. The coordinates are as shown:

No constraint is redundant if

$S_{13}$ is between $(4,0)$ and $S_{12}$, that is,

$$0 < 3\Delta - 6 < 12\beta \quad \text{or} \quad 2 < \Delta < 2 + 4\beta$$

and $S_{23}$ is between $(0,12\beta)$ and $S_{12}$, that is,

$$0 < 2\Delta - 8\beta < 4 \quad \text{or} \quad 4\beta < \Delta < 2 + 4\beta$$

$\Rightarrow \ \max \{2, 4\beta\} < \Delta < 2 + 4\beta$

$\Rightarrow \ 7D$
Q8: Let \( r_1, r_2, r_3 \) be the three ratios of which \( \min (r_j, r_j \geq 0) \) determines the leaving variable. For \( x_5 = 12\beta \) it appears from SF that

\[ r_1 \text{ is irrelevant since the coefficient to } x_4 \text{ is positive,} \]

\[ r_2 = 12\beta/\beta = 12 \]

\[ r_3 = (4 - (2\Delta - 8\beta))/2(\beta/3) = 12 + (3/\beta)(2 - \Delta) \]

Since no constraint is redundant, \( \max (2, 4\beta) < \Delta \), cf. Q7, that is, \( \Delta \) must be strictly greater than 2. Hence, \( r_3 < r_2 \)

\[ \Rightarrow 8C \]

Q9 (text question):

9.1 True. With no redundant constraints, the point \( S_{12} \) defined above is outside the feasible region for all values of \( \beta, \Delta \). Either \( x_4 \) or \( x_5 \) must therefore be strictly positive.

9.2 False. \( x_3 = x_5 = 0 \) at \( S_{13} \).

9.3 True. We know from 9.1 that \( P(\alpha, \beta, \Delta) \) has no solution in which all three constraints hold as equations. Positive slack in an optimal solution to \( P(\alpha, \beta, \Delta) \) implies that the corresponding dual variable is zero, cf. Complementary slackness.

9.4 False. The correct sum of the dual variables is provided in

9.5 True. \( y_2^0 \) and \( y_3^0 \) are the coefficients (with opposite sign) to \( x_4 \) and \( x_5 \), respectively, in the \( z \)-equation of SF. Note that \( y_1^0 = 0 \) since \( x_3 \) is a basic variable.

Max flow (Q10)

Q10 (text question):

Note: Here, \( c(v_i, v_j) = 2j - i \)
The figure shows \( G_6 \) and an optimal solution (for each edge: flow/capacity) found by inspection. Value of max flow = 3+6+11 = 20.

Min cut: \( S = \{v_1,v_4\}, \ T = V \setminus S \). Capacity of \( (S,T) \) = 20.

10.2 The vertex set \( V \) of \( G_{2n} \) can be partitioned into two subsets \( X \) and \( Y \),

\[
X = \{v_i, \ i \text{ is odd}\}, \quad Y = \{v_i, \ i \text{ is even}\}
\]

All edges of \( G_{2n} \) connect vertices belonging to different sets \( \Rightarrow \) the number of edges in an "XX"-path or "YY"-path must be even. All other paths must comprise an odd number of edges. Finally, to complete the proof we note that \( v_1 \in X \) and \( v_{2n} \in Y \) for all values of \( n \).

Appendix

Whether the slack form or the tableau form best illustrates the basic ideas of the simplex algorithm is a matter of taste.

For the sake of comparison, however, most of the computations made on our tour de force with Q1-Q9 are summarized here:

<table>
<thead>
<tr>
<th>Equation</th>
<th>BV</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( -z )</th>
<th>RHS</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>( x_3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>4</td>
<td>⬤</td>
<td></td>
</tr>
<tr>
<td>Q</td>
<td>( x_4 )</td>
<td>1/\beta</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>12</td>
<td>12\beta</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>( x_5 )</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td>6\Delta</td>
<td>3\Delta</td>
<td></td>
</tr>
</tbody>
</table>

\[
z = z 3 a 1 0
\]

<table>
<thead>
<tr>
<th>Equation</th>
<th>BV</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( -z )</th>
<th>RHS</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>( x_3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Q=\beta Q</td>
<td>( x_2 )</td>
<td>1</td>
<td>( \beta )</td>
<td></td>
<td></td>
<td></td>
<td>12\beta</td>
<td>⬤</td>
<td></td>
</tr>
<tr>
<td>R=R-2Q</td>
<td>( x_5 )</td>
<td>3</td>
<td>-2\beta</td>
<td>1</td>
<td></td>
<td></td>
<td>6\Delta-2\beta</td>
<td>2\Delta-8\beta</td>
<td></td>
</tr>
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</table>

\[
z = z 3 -z \alpha 1 -12\alpha \beta
\]

<table>
<thead>
<tr>
<th>Equation</th>
<th>BV</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( -z )</th>
<th>RHS</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>R=R/3</td>
<td>( x_4 )</td>
<td>1</td>
<td>-2\beta/3</td>
<td>1/3</td>
<td></td>
<td></td>
<td>2\Delta-8\beta</td>
<td>( r_1 )</td>
<td>⬤</td>
</tr>
<tr>
<td>Q</td>
<td>( x_2 )</td>
<td>1</td>
<td>( \beta )</td>
<td></td>
<td></td>
<td></td>
<td>12\beta</td>
<td>( r_2 )</td>
<td></td>
</tr>
<tr>
<td>P=P-R</td>
<td>( x_5 )</td>
<td>1</td>
<td>2\beta/3</td>
<td>-1/3</td>
<td></td>
<td></td>
<td>4-(2\Delta-8\beta)</td>
<td>( r_3 )</td>
<td></td>
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</tbody>
</table>

\[
z = z 3R -z 0 (2-\alpha)\beta -1 1 -(6\Delta+12\beta(\alpha-2))
\]

\[\uparrow \quad \uparrow \quad \uparrow \]

\[-y^0_1 \quad -y^0_2 \quad -y^0_3 \]
Though not needed for answering Q6 and Q9, $P(\alpha, \beta, \Delta)$ and $D(\alpha, \beta, \Delta)$ are shown here in full:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>rel</th>
<th>RHS</th>
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<tbody>
<tr>
<td>$y_1$</td>
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<td></td>
<td></td>
<td>$\leq$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$1/\beta$</td>
<td></td>
<td></td>
<td>$\leq$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>3</td>
<td>2</td>
<td></td>
<td>$\leq$</td>
</tr>
</tbody>
</table>

\[
\text{rel} \quad \geq \quad \geq \quad 3 \quad \alpha
\]

\[
\min w = 4y_1 + 12y_2 + 6\Delta y_3
\]

\[
y_1 + (1/\beta)y_2 + 3y_3 \geq 3
\]

\[
2y_3 \geq \alpha
\]

\[
y_1, y_2 \geq 0
\]