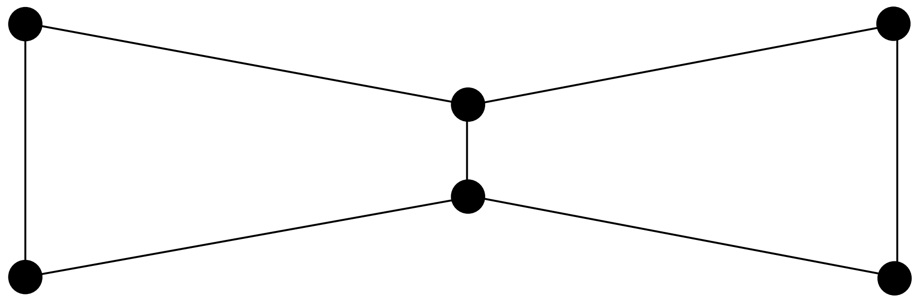
Minimum-Length Two-Connected Networks



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Outline

- Motivation and background
- Minimum-length two-connected (Steiner) networks
- Exchanging subgraphs
- Vertex degree properties
- Chord-path properties
- Cycle properties
- Smallest networks with Steiner vertices

Motivation and Background

Main application: Design of communication networks

Task: Construct a network that

- has low *cost*
- fulfills certain *connectivity* requirements
- offers *survivability* against network failures

Assumptions:

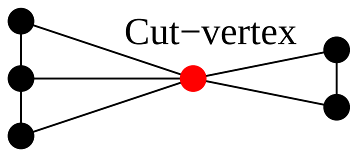
1. Cost is associated with the links of the network.
2. All vertices should be able to communicate with each other (but no bandwidth/flow requirements).
3. The network should offer protection against single *link* or single *vertex* failures.

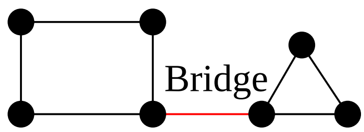
Basic Definitions

Given a connected undirected graph $G = (V, E)$.

Cut vertex
Vertex whose removal increases the number of components.

Bridge
Edge whose removal increases the number of components.





G is **two-vertex connected**: has *no* cut-vertex
G is **two-edge connected**: has *no* bridge

Observation: Two-vertex connected $\Rightarrow$ Two-edge connected

Basic Results

Equivalent statements for a **two-vertex** connected graph G:

(i) G is two-vertex connected

(ii) There are two vertex disjoint paths between every pair of vertices in G

(iii) Every two vertices (or edges) of G lie on a common cycle

Equivalent statements for a **two-edge** connected graph G:

(i) G is two-edge connected

(ii) There are two edge disjoint paths between every pair of vertices in G

(iii) Every two vertices (or edges) of G lie on a common circuit

Two-Connected Network Problem

Given

- a set V of vertices (e.g., points in the plane)
- a metric d on V (e.g., the Euclidean metric)

Find a subset of edges $E \subseteq V \times V$ such that

- $G = (V, E)$ is *two-connected*, and
- $d(E) = \sum_{(u,v)\in E} d(u, v)$ is *minimized*.

Note: It is equivalent to find a minimum-length two-vertex connected and two-edge connected network for a metric distance function (will be shown later).

NP-hard problem even for Euclidean distances in the plane [Krznaric et al., 1997].

Two-Connected Steiner Network Problem

Given

- a set V of vertices (e.g., points in the plane)
- a metric d on V (e.g., the Euclidean metric)
- a set of terminals $Z \subseteq V$

Find a subset of vertices $W \subseteq V$ and edges $E \subseteq W \times W$ such that

- $Z \subseteq W$ (all terminals are spanned),
- $G = (W, E)$ is *two-connected*, and
- $d(E) = \sum_{(u,v) \in E} d(u, v)$ is *minimized*.

Generalization of the minimum-length two-connected network problem and therefore also NP-hard.

Relation to Metric TSP

Observation: Hamilton cycle through V is a valid solution.

$TSP(V)$: Minimum-length cycle through V

$TC(V)$: Minimum-length two-connected network for V

$STC(Z, V)$: Minimum-length two-connected Steiner network for $Z \subseteq V$

Theorem [Monma et al., 1990]

$$\frac{d(TSP(V))}{d(TC(V))} \leq \frac{4}{3}$$

and

$$\frac{d(TC(Z))}{d(STC(Z, V))} \leq \frac{4}{3}$$

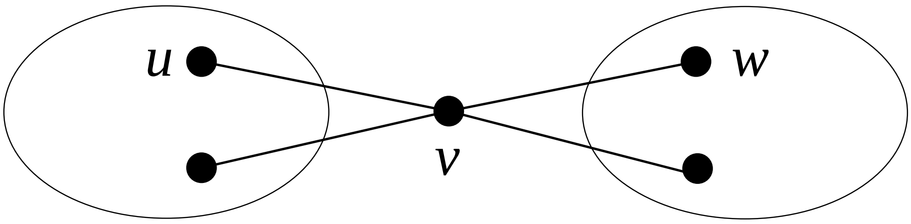
Two-Vertex and Two-Edge Connected Networks

Lemma
The length of minimum-length two-vertex and two-edge connected networks are identical for *metric* distance functions.

Proof
Let $d(TC^v)$ and $d(TC^e)$ be the minimum length of a two-vertex and two-edge connected network, respectively.

Clearly $d(TC^v) \geq d(TC^e)$ since a two-vertex connected network is always two-edge connected.

Assume we have a two-edge connected network which is *not* two-vertex connected:



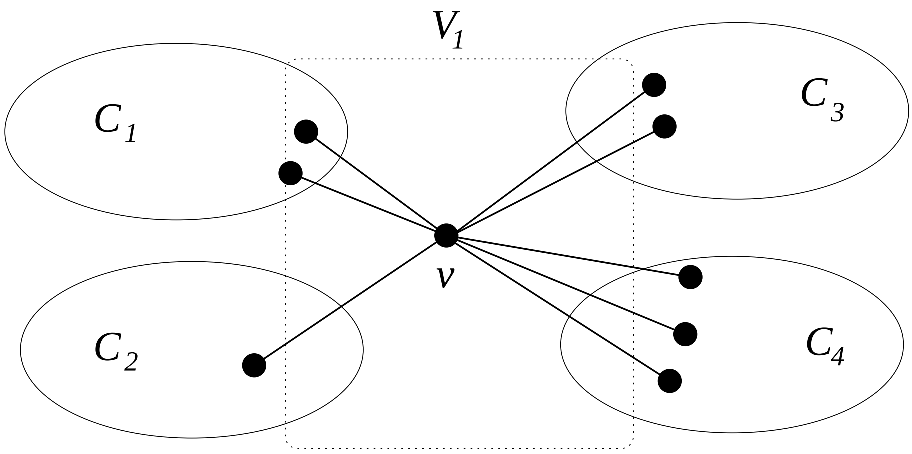
Replace (u, v) and (v, w) with (u, w); length will not increase and vertex v is no longer a cut vertex.

Exchanging Subgraphs

Given a two-connected graph $G = (V, E)$ it is possible to make simple local exchanges that preserve two-connectedness.

Lemma (“exchange lemma”)
Let $G_1 = (V_1, E_1)$ be a subgraph of $G = (V, E)$ induced by $V_1 \subseteq V$. Then replacing E_1 in G by a collection of edges E_2 such that $G_2 = (V_1, E_2)$ is two-connected, results in a graph on V which is two-connected.

Proof
Let $G^* = (V, (E \setminus E_1) \cup E_2)$ be the result of the local exchange, and suppose that G^* is not two-connected.



Let vertex v be a cut vertex. Each component $C_1, \ldots, C_k$ must intersect V_1 (otherwise G was not two-connected). Since G_2 is two-connected there exists a path between any two components C_i and C_j that avoids v. Thus v is *not* a cut vertex.

Vertex Degree Properties

Technical assumption: We seek minimum-length two-connected networks for which the **total degree** of all vertices is minimized. This avoids trivial degeneracies.

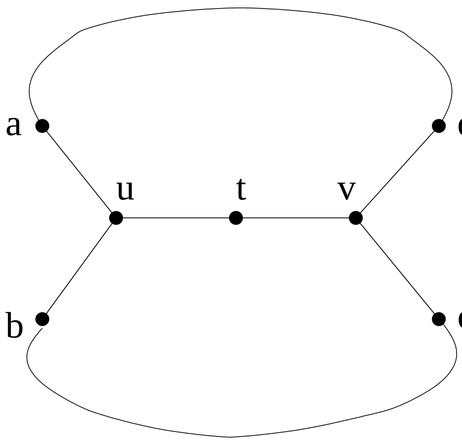
Let SMN denote an arbitrary minimum-length two-connected network for which the total degree is minimized.

Theorem [Monma et al., 1990]
Every vertex in a SMN has degree *2 or 3*. Every Steiner vertex (in $W \setminus Z$) has degree exactly *3*.

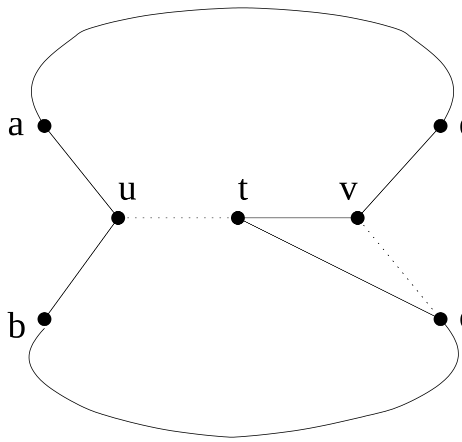
Chord-path Properties I

Chord-path for a cycle C: Path in SMN between two distinct vertices u and v on C that — except from u and v — shares neither vertices nor edges with C.

Lemma
Any chord-path in a SMN must have at least *three* edges.



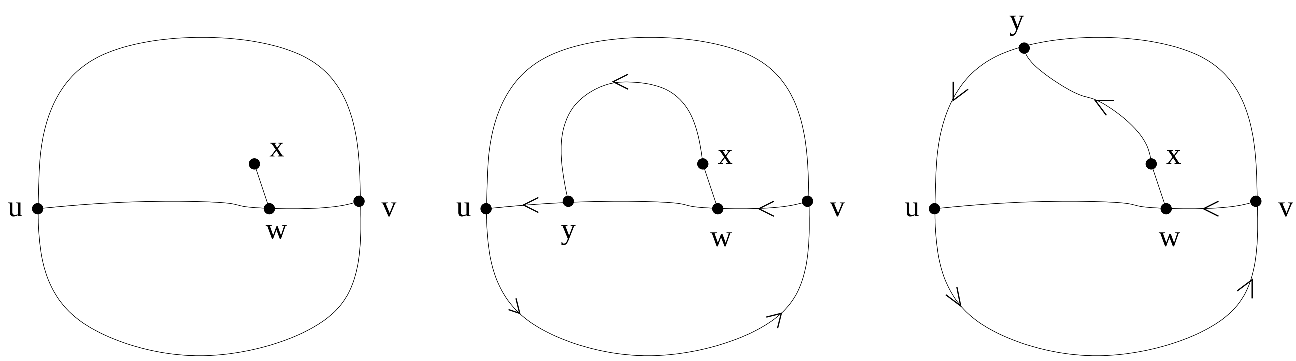
a)



b)

Chord-path Properties II

Lemma
Any chord-path in a SMN must have a pair of consecutive terminals of degree 2 in its interior.



a) b) c)

Cycle Properties

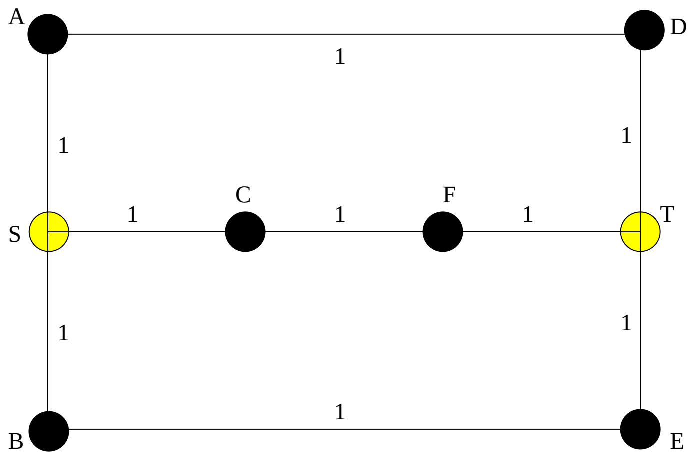
Theorem
Consider a cycle C that contains a vertex of degree 3 in a SMN. Then C must have *two* pairs of consecutive terminals, both of degree 2 in SMN. Furthermore, these two terminal pairs must be separated on C by a pair of vertices of degree 3.

Corollary
If the total number of terminals in a SMN is at least four, then every cycle in SMN has at least *four* terminals of degree 2.

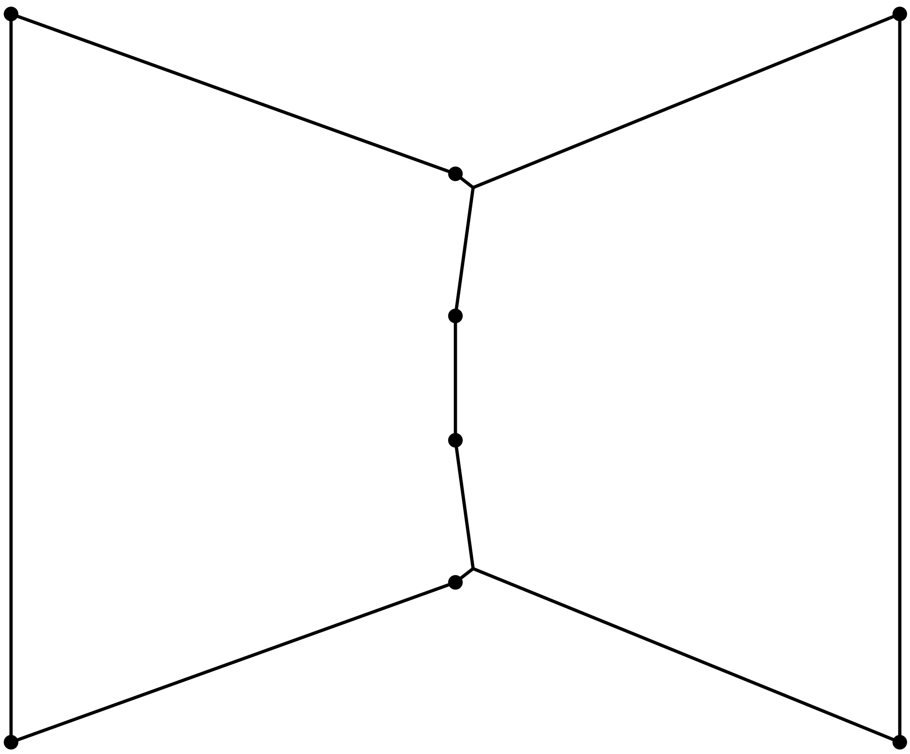
Smallest Networks with Steiner Vertices

Any SMN that has at least one Steiner vertex *must* have at least *6* terminals.

Tight example:



Smallest Known Euclidean Network with Steiner Vertices



This SMN has 8 terminals. Does a network with 6 or 7 terminals exist?