Introduction to Optimization:

Written Exam, June 2004

Multiple-choice knapsack problem

We are given a number of classes $N_1, \dots, N_k$ of items. Each item $j \in N_i$ has an associated profit p_{ij} and a weight w_{ij}. The objective of the problem is to choose exactly one item from each class N_i such that the profit sum of the chosen items is maximized, while the weight sum of the chosen items cannot exceed a given capacity c.

In the following instance we have $k = 3$ classes, and the capacity is $c = 9$.

$N_1 = \{1, 2, 3\}$ $N_2 = \{1, 2\}$ $N_3 = \{1, 2, 3\}$

j	1	2	3
$p_{1,j}$	0	4	6
$w_{1,j}$	0	3	4

j	1	2
$p_{2,j}$	2	3
$w_{2,j}$	1	2

j	1	2	3
$p_{3,j}$	0	3	4
$w_{3,j}$	3	4	8

Q11: Solve the above problem to integer optimality. What is the optimal solution value z?

11A) $z = 8$
11B) $z = 9$
11C) $z = 10$
11D) $z = 11$
11E) $z = 12$
11F) $z = 13$

Q12: We introduce the binary variables x_{ij} to indicate if item j is chosen in class N_i. What is the correct formulation of the multiple-choice knapsack problem?

12A)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j \in N_i} p_{ij} x_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j \in N_i} w_{ij} x_{ij} \le c \\ & \textstyle\sum_{j \in N_i} x_{ij} = 1, && i = 1, \dots, k \\ & x_{ij} \in \{0, 1\}, && i = 1, \dots, k, j \in N_i \end{aligned}$$

12B)
$$\begin{aligned} \max\ & \textstyle\sum_{j \in N_i} p_{ij} x_{ij}, && i = 1, \dots, k \\ \text{s.t.}\ & \textstyle\sum_{j \in N_i} w_{ij} x_{ij} \le c, && i = 1, \dots, k \\ & x_{ij} = 1, && i = 1, \dots, k, j \in N_i \\ & x_{ij} \in \{0, 1\}, && i = 1, \dots, k, j \in N_i \end{aligned}$$

12C)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j \in N_i} p_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j \in N_i} w_{ij} \le c \\ & \textstyle\sum_{j \in N_i} x_{ij} = 1, && i = 1, \dots, k \\ & x_{ij} \in \{0, 1\}, && i = 1, \dots, k, j \in N_i \end{aligned}$$

12D)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j \in N_i} p_{ij} x_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j \in N_i} w_{ij} x_{ij} \le c \\ & \textstyle\sum_{i=1}^{k} x_{ij} = 1, && j \in N_i \\ & x_{ij} \in \{0, 1\}, && i = 1, \dots, k, j \in N_i \end{aligned}$$

12E)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j=1}^{k} p_{ij} x_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j=1}^{k} w_{ij} x_{ij} \le c \\ & \textstyle\sum_{i=1}^{k} x_{ij} = 1, && j \in N_i \\ & x_{ij} \in \{0, 1\}, && i = 1, \dots, k, j \in N_i \end{aligned}$$

12F)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j \in N_i} p_{ij} x_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j \in N_i} w_{ij} x_{ij} \le c \\ & \textstyle\sum_{i=1}^{k} \sum_{j \in N_i} x_{ij} = 1 \\ & x_{ij} \in \{0, 1\}, && i = 1, \dots, k, j \in N_i \end{aligned}$$

In order to tighten the formulation we would like to derive a cover. A cover $C = \{h_1, \ldots, h_k\}$ consists of one index h_i from each class N_i such that

$$\sum_{i=1}^{k} w_{ih_i} > c$$

If C is a cover, we may impose a cover inequality of the form

$$\sum_{i=1}^{k} x_{ih_i} \leq d$$

Q13: What is the smallest value of d such that the above cover inequality is valid?

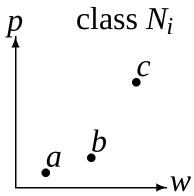
13A) $d = c$
13B) $d = |N_i|$
13C) $d = k$
13D) $d = c - 1$
13E) $d = |N_i| - 1$
13F) $d = k - 1$

Q14: **(text question)** If the LP-solution (x^*_{ij}) to the multiple-choice knapsack problem is non-integral, describe a separation algorithm which separates the most violated cover inequality. (*Hint:* See Wolsey Section 9.3.3 for the "normal" cover inequalities).

Q15: **(text question)** Prove that if we have three items a, b and c in the same class N_i such that $w_{ia} \leq w_{ib} \leq w_{ic}$ and

$$\frac{p_{ic} - p_{ia}}{w_{ic} - w_{ia}} \geq \frac{p_{ib} - p_{ia}}{w_{ib} - w_{ia}}$$

then an optimal LP-solution exists where $x_{ib} = 0$. (See the following example for a geometrical interpretation).



j	a	b	c
p_{ij}	1	2	7
w_{ij}	2	5	8

Integer programming

Consider the following problem

$$\begin{array}{llcrl}
\text{maximize} & x_1 + 2x_2 & & & \\
\text{subject to} & 4x_1 + x_2 & \leq & 8 & \text{(a)} \\
 & x_1 - x_2 & \geq & -1 & \text{(b)} \\
 & x_1 & \leq & 2 & \text{(c)} \\
 & x_2 & \leq & 2 & \text{(d)} \\
 & x_1 & \geq & 0 & \text{(e)} \\
 & x_2 & \geq & 0 & \text{(f)} \\
 & x_1, x_2 \in \mathbb{Z} & & &
\end{array}$$

Q16: What is the largest set of facet-defining inequalities?

16A) all of the constraints (a) to (f)
16B) (d), (e), (f)
16C) (b), (d), (e), (f)
16D) (c), (d), (e), (f)
16E) (b)
16F) (b), (e), (f)

Q17: Assume that we solve the problem to LP-optimality. What is the value of the dual variables y_a, y_d corresponding to constraints (a) and (d).

17A) $y_a = \frac{1}{2}, y_d = \frac{3}{7}$
17B) $y_a = 2, y_d = 0$
17C) $y_a = \frac{1}{4}, y_d = \frac{2}{3}$
17D) $y_a = \frac{1}{4}, y_d = \frac{7}{4}$
17E) $y_a = \frac{1}{3}, y_d = \frac{4}{3}$
17F) $y_a = \frac{1}{4}, y_d = \frac{1}{3}$

Q18: Assume that we Lagrangian relax constraints (a) and (b) using multipliers λ_a and λ_b respectively. What is the optimal value of the Lagrangian multipliers when solving the Lagrangian dual?

18A) $\lambda_a = \frac{1}{2}, \lambda_b = 2$
18B) $\lambda_a = \frac{1}{2}, \lambda_b = 0$
18C) $\lambda_a = \frac{1}{3}, \lambda_b = \frac{1}{4}$
18D) $\lambda_a = \frac{1}{2}, \lambda_b = 3$
18E) $\lambda_a = \frac{1}{4}, \lambda_b = 0$
18F) $\lambda_a = \frac{1}{4}, \lambda_b = 3$

Model Building

An air cargo company is planning the transportation of some goods from city A to city B. Every plane can carry at most q weight units. A number of different items $j = 1, \ldots, n$, need to be transported, item j taking up w_j weight units. Each plane $i = 1, \ldots, n$ should contain at least 4 different items to ensure proper balancing. The air cargo company wishes to minimize the number of planes used.

Q19: The problem is formulated as an integer-programming model, in which $y_i = 1$ if and only if plane i is used, and $x_{ij} = 1$ if and only if item j is sent by plane i. Which of the following models is a correct formulation of the problem

19A)
$$\begin{aligned}&\min \textstyle\sum_{i=1}^n y_i\\ \text{s.t. } &\textstyle\sum_{j=1}^n w_j x_{ij} \le y_i q, && i=1,\dots,n\\ &\textstyle\sum_{j=1}^n x_{ij} \ge 4, && i=1,\dots,n\\ &\textstyle\sum_{i=1}^n x_{ij} = 1, && j=1,\dots,n\\ &x_{ij}, y_i \in \{0,1\}, && i,j=1,\dots,n\end{aligned}$$

19D)
$$\begin{aligned}&\min \textstyle\sum_{i=1}^n y_i\\ \text{s.t. } &\textstyle\sum_{i=1}^n w_j x_{ij} \le q\\ &\textstyle\sum_{i=1}^n x_{ij} \ge 4, && i=1,\dots,n\\ &\textstyle\sum_{i=1}^n x_{ij} = 1, && j=1,\dots,n\\ &x_{ij}, y_i \in \{0,1\}, && i,j=1,\dots,n\end{aligned}$$

19B)
$$\begin{aligned}&\min \textstyle\sum_{i=1}^n y_i\\ \text{s.t. } &\textstyle\sum_{j=1}^n w_j x_{ij} \le y_i q, && i=1,\dots,n\\ &\textstyle\sum_{j=1}^n x_{ij} \ge 4, && i=1,\dots,n\\ &\textstyle\sum_{j=1}^n x_{ij} = 1, && i=1,\dots,n\\ &x_{ij}, y_i \in \{0,1\}, && i,j=1,\dots,n\end{aligned}$$

19E)
$$\begin{aligned}&\min \textstyle\sum_{i=1}^n y_i\\ \text{s.t. } &\textstyle\sum_{i=1}^n w_j x_{ij} \le q\\ &\textstyle\sum_{i=1}^n x_{ij} \ge 4, && i=1,\dots,n\\ &\textstyle\sum_{j=1}^n x_{ij} = 1, && i=1,\dots,n\\ &x_{ij}, y_i \in \{0,1\}, && i,j=1,\dots,n\end{aligned}$$

19C)
$$\begin{aligned}&\min \textstyle\sum_{i=1}^n y_i\\ \text{s.t. } &\textstyle\sum_{j=1}^n w_j x_{ij} \le y_i q, && i=1,\dots,n\\ &\textstyle\sum_{i=1}^n x_{ij} = 1, && j=1,\dots,n\\ &x_{ij}, y_i \in \{0,1\}, && i,j=1,\dots,n\end{aligned}$$

19F)
$$\begin{aligned}&\min y_i\\ \text{s.t. } &\textstyle\sum_{i=1}^n w_j x_{ij} \le q\\ &\textstyle\sum_{i=1}^n x_{ij} \ge 4, && i=1,\dots,n\\ &\textstyle\sum_{i=1}^n x_{ij} = 1, && j=1,\dots,n\\ &x_{ij}, y_i \in \{0,1\}, && i,j=1,\dots,n\end{aligned}$$

Q20: For security reasons, it is decided that either items 1 and 2 should go by the same plane, or items 1, 2, 3 should go by different planes. What is the correct integer linear formulation of the constraints?

20A)
$$\begin{aligned}&x_{i1}-x_{i2}=0 && i=1,\dots,n\\ &x_{i1}+x_{i2}\le 1 && i=1,\dots,n\\ &x_{i1}+x_{i3}\le 1 && i=1,\dots,n\\ &x_{i2}+x_{i3}\le 1 && i=1,\dots,n\\ &\delta\in\{0,1\}\end{aligned}$$

20D)
$$\begin{aligned}&x_{i1}-x_{i2}+\delta\le 1 && i=1,\dots,n\\ &x_{i1}-x_{i2}-\delta\ge -1 && i=1,\dots,n\\ &x_{i1}+x_{i2}+(1-\delta)\le 2 && i=1,\dots,n\\ &x_{i1}+x_{i3}+(1-\delta)\le 2 && i=1,\dots,n\\ &x_{i2}+x_{i3}+(1-\delta)\le 2 && i=1,\dots,n\\ &\delta\in\{0,1\}\end{aligned}$$

20B)
$$\begin{aligned}&\delta(x_{i1}-x_{i2})=0 && i=1,\dots,n\\ &(1-\delta)(x_{i1}+x_{i2}\le 1) && i=1,\dots,n\\ &(1-\delta)(x_{i1}+x_{i3}\le 1) && i=1,\dots,n\\ &(1-\delta)(x_{i2}+x_{i3}\le 1) && i=1,\dots,n\\ &\delta\in\{0,1\}\end{aligned}$$

20E)
$$\begin{aligned}&x_{i1}-x_{i2}+\delta\le 1 && i=1,\dots,n\\ &x_{i1}-x_{i2}-\delta\ge -1 && i=1,\dots,n\\ &x_{i1}+x_{i2}+\delta\le 2 && i=1,\dots,n\\ &x_{i1}+x_{i3}+\delta\le 2 && i=1,\dots,n\\ &x_{i2}+x_{i3}+\delta\le 2 && i=1,\dots,n\\ &\delta\in\{0,1\}\end{aligned}$$

20C)
$$\begin{aligned}&x_{i1}-x_{i2}\le\delta && i=1,\dots,n\\ &x_{i1}-x_{i2}\ge\delta && i=1,\dots,n\\ &x_{i1}+x_{i2}+\le 2-\delta && i=1,\dots,n\\ &x_{i1}+x_{i3}+\le 2-\delta && i=1,\dots,n\\ &x_{i2}+x_{i3}+\le 2-\delta && i=1,\dots,n\\ &\delta\in\{0,1\}\end{aligned}$$

20F)
$$\begin{aligned}&x_{i1}-x_{i2}+\delta_1\le 1 && i=1,\dots,n\\ &x_{i1}-x_{i2}-\delta_1\le -1 && i=1,\dots,n\\ &x_{i1}+x_{i2}+\delta_2\le 2 && i=1,\dots,n\\ &x_{i1}+x_{i3}+\delta_2\le 2 && i=1,\dots,n\\ &x_{i2}+x_{i3}+\delta_2\le 2 && i=1,\dots,n\\ &\delta_1,\delta_2\in\{0,1\}\end{aligned}$$

THE END

Answers

Q11: In class N_3, the first item can be removed since it has profit $p_{3,1} = 0$. The last item in the same class can also be removed since if we choose $x_{3,3} = 1$ then the only other solution is $x_{1,1} = x_{2,1} = 1$, having a too small profit sum.

Knowing that the second item from class 3 should be chosen, we find by inspection that the optimal solution is $x_{1,3} = x_{2,1} = x_{3,2} = 1$ with all other variables set to zero. This gives the objective value $z = 11$. Hence, answer 11D) is correct.

Q12: We should maximize the profit sum

$$\sum_{i=1}^{k} \sum_{j \in N_i} p_{ij} x_{ij}$$

Exactly one item should be chosen from each class

$$\sum_{j \in N_i} x_{ij} = 1 \quad i = 1, \ldots, k$$

The sum of the chosen items may not exceed the capacity c

$$\sum_{i=1}^{k} \sum_{j \in N_i} w_{ij} x_{ij} \leq c$$

The decision variables are binary

$$x_{ij} \in \{0, 1\} \quad i = 1, \ldots, k, j \in N_i$$

Hence, the correct formulation is 12A).

Q13: Since not all items in C can be in the same knapsack, at most $|C| - 1 = k - 1$ of the items can be chosen. Hence, 13F) is the correct answer.

Q14: This is a straightforward generalization of the separation algorithm for normal cover inequalities.

Q15: Since a convex combination of items a and c gives at least as good a profit as item b, the latter item will never be used in an LP-solution.

More formally, if $x_{i,b} > 0$ then solving

$$x_{ia} + x_{ic} = x_{ib}$$
$$w_{ia} x_{ia} + w_{ic} x_{ic} = w_{ib} x_{ib}$$

gives a convex combination of item a and c with the same weight contribution as item b. The solution to the above set of equations is

$$x_{ic} = x_{ib} \frac{w_{ib} - w_{ia}}{w_{ic} - w_{ia}}$$
$$x_{ia} = x_{ib} \frac{w_{ic} - w_{ib}}{w_{ic} - w_{ia}}$$

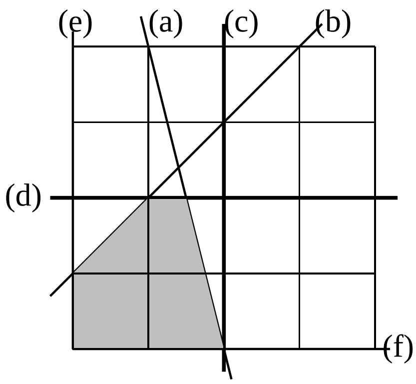
The profit sum satisfies

$$p_{ia}x_{ia} + p_{ic}x_{ic} \geq p_{ib}x_{ib} \Leftrightarrow$$
$$p_{ia}x_{ib}\tfrac{w_{ic}-w_{ib}}{w_{ic}-w_{ia}} + p_{ic}x_{ib}\tfrac{w_{ib}-w_{ia}}{w_{ic}-w_{ia}} \geq p_{ib}x_{ib} \Leftrightarrow$$
$$p_{ia}(w_{ic}-w_{ib}) + p_{ic}(w_{ib}-w_{ia}) \geq p_{ib}(w_{ic}-w_{ia})$$

since from the stated we have

$$\tfrac{p_{ic}-p_{ia}}{w_{ic}-w_{ia}} \geq \tfrac{p_{ib}-p_{ia}}{w_{ib}-w_{ia}} \Leftrightarrow$$
$$(p_{ic}-p_{ia})(w_{ib}-w_{ia}) \geq (p_{ib}-p_{ia})(w_{ic}-w_{ia}) \Leftrightarrow$$
$$p_{ic}(w_{ib}-w_{ia}) - p_{ia}(w_{ib}-w_{ia}) \geq p_{ib}(w_{ic}-w_{ia}) - p_{ia}(w_{ic}-w_{ia}) \Leftrightarrow$$
$$p_{ia}(w_{ic}-w_{ib}) + p_{ic}(w_{ib}-w_{ia}) \geq p_{ia}(w_{ic}-w_{ia})$$

Q16: The problem is defined in two variables so we can draw it in the plane as follows



It is easily seen that inequalities (b), (e), (f) are facet-defining. Hence, answer 16F) is correct.

Q17: The LP-relaxed problem is

$$\begin{array}{llrl}
\max & x_1 + 2x_2 & & \\
\text{s.t.} & 4x_1 + x_2 \leq 8 & & \text{(a)} \\
& -x_1 + x_2 \leq 1 & & \text{(b)} \\
& x_1 \leq 2 & & \text{(c)} \\
& x_2 \leq 2 & & \text{(d)} \\
& -x_1 \leq 0 & & \text{(e)} \\
& -x_2 \leq 0 & & \text{(f)} \\
& x_1, x_2 \in \mathbb{R} & &
\end{array}$$

the dual problem becomes

$$
\begin{array}{lllllllll}
\min & 8y_a & + \ y_b & + \ 2y_c & + \ 2y_d & & & \\
\text{s.t.} & 4y_a & - \ y_b & + \ \ y_c & & - \ y_e & & \geq 1 \\
& y_a & + \ y_b & & + \ \ y_d & & - \ y_f & \geq 2 \\
& \multicolumn{7}{l}{y_a, y_b, y_c, y_d, y_e, y_f \in \mathbb{R}}
\end{array}
$$

The primal solution is $x_1 = \frac{3}{2}$, $x_2 = 2$ with objective value $z = \frac{3}{2} + 2 \cdot 2 = 5\frac{1}{2}$. From complementary slackness we get:

$$
\begin{aligned}
&4y_a - y_b + y_c - y_e = 1 \\
&y_a + y_b + y_d - y_f = 2 \\
&y_b = 0 \\
&y_c = 0 \\
&y_e = 0 \\
&y_f = 0
\end{aligned}
$$

Giving the equations

$$
\begin{aligned}
&4y_a = 1 \\
&y_a + y_d = 2
\end{aligned}
$$

with optimal solution $y_a = \frac{1}{4}$, $y_d = \frac{7}{4}$, and solution value $z = 8 \cdot \frac{1}{4} + 2 \cdot \frac{7}{4} = 5\frac{1}{2}$. Hence, answer 17D) is correct.

Q18: Lagrangian relaxing constraints (a) and (b) we notice that the remaining constraints define the convex hull of the integer solutions to (c), (d), (e) and (f). Hence, the optimal choice of Lagrangian multipliers λ_a, λ_b correspond to the dual variables y_a, y_b. Hence, $\lambda_a = \frac{1}{4}, \lambda_b = 0$. The correct answer is then 18E).

Q19: We should minimize the number of planes, i.e.

$$\sum_{i=1}^{n} y_i$$

If plane i is used (i.e. $y_i = 1$) then the sum of the weights should not exceed the plane capacity q. If the plane is not used (i.e. $y_i = 0$) then no capacity is available in this plane.

$$\sum_{j=1}^{n} w_j x_{ij} \leq y_i q$$

There should be at least 4 different items in each plane i

$$\sum_{j=1}^{n} x_{ij} \geq 4$$

All items should go into some plane

$$\sum_{i=1}^{n} x_{ij} = 1$$

Finally, all decision variables are binary, hence formulation 19A) is correct.

Q20: We use $\delta = 1$ to indicate that items 1 and 2 go by the same plane, and $\delta = 0$ to indicate that items 1, 2, 3 go by different planes.

The first constraint is mathematically formulated as

$$\delta = 1 \Rightarrow x_{i1} - x_{i2} = 0$$

for each value of $i = 1, \ldots, n$, which in linear form becomes

$$\begin{aligned} x_{i1} - x_{i2} + \delta &\leq 1 \\ x_{i1} - x_{i2} - \delta &\geq -1 \end{aligned}$$

The latter constraint is mathematically formulated as

$$\delta = 0 \Rightarrow (x_{i1} + x_{i2} \leq 1) \wedge (x_{i1} + x_{i3} \leq 1) \wedge (x_{i2} + x_{i3} \leq 1)$$

for each value of $i = 1, \ldots, n$, which in linear form becomes

$$\begin{aligned} x_{i1} + x_{i2} + (1 - \delta) &\leq 2 \\ x_{i1} + x_{i3} + (1 - \delta) &\leq 2 \\ x_{i2} + x_{i3} + (1 - \delta) &\leq 2 \end{aligned}$$

Hence, answer 20D) is correct.