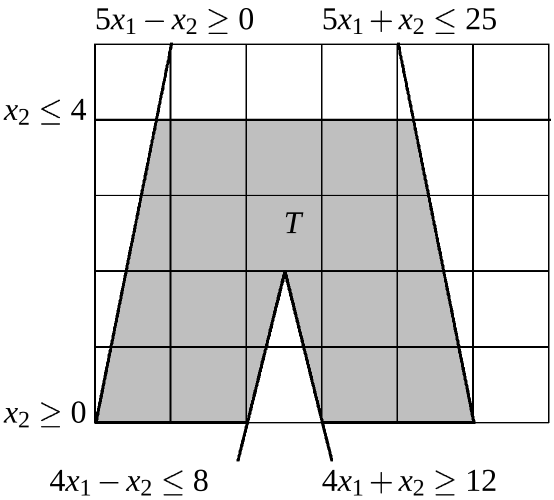
Introduction to Optimization:

Written Exam, December 2003

Chippendales

In order to meet the critics of several female students saying that beer does not appeal to women, Tuberg starts to support shows with the Chippendales. The trunks used by Chippendales are designed according to a secret cut T depicted at the following figure:



Q11: To model T as a MIP model the following constraints are obviously necessary

$$
\begin{aligned}
x_2 &\leq 4 \\
x_2 &\geq 0 \\
5x_1 - x_2 &\geq 0 \\
5x_1 + x_2 &\leq 25
\end{aligned}
$$

Which constraints should be added to the model to get a complete integer linear model of T?

A)
$$4x_1 - x_2 + 12\delta \le 20$$
$$4x_1 + x_2 - 12\delta \ge 0$$
$$\delta \in \{0,1\}$$

B)
$$4x_1 - x_2 + 12\delta_A \le 20$$
$$4x_1 + x_2 - 12\delta_B \ge 0$$
$$\delta_A + \delta_B \ge 1$$
$$\delta_A, \delta_B \in \{0,1\}$$

C)
$$4x_1 - x_2 \le 8$$
$$4x_1 + x_2 \ge 12$$

D)
$$x_2 + 4|x_1 - \tfrac{5}{2}| \ge 2$$

E)
$$4x_1 - x_2 + 12\delta_A \le 20$$
$$4x_1 + x_2 - 12\delta_B \ge 0$$
$$\delta_A + \delta_B \le 1$$
$$\delta_A, \delta_B \in \{0,1\}$$

F)
$$\delta_A(4x_1 - x_2 \le 8)$$
$$\delta_B(4x_1 + x_2 \ge 12)$$
$$\delta_A + \delta_B \ge 1$$
$$\delta_A, \delta_B \in \{0,1\}$$

Q12: In order to improve sales, the management of Tuberg decides to change the design of the trunks such that it becomes $T' = \text{conv}\{x : x \in T, x \in \mathbb{N}^2\}$. The members of Chippendales are asked to write up an LP-model describing T'. Despite their talents the members end up having written six different models. Which of the formulations is a correct LP-model describing T'?

A)
$$x_2 \le 4$$
$$x_2 \ge 0$$
$$4x_1 - x_2 \ge 0$$
$$4x_1 + x_2 \le 20$$
$$4x_1 - x_2 + 12\delta_A \le 20$$
$$4x_1 + x_2 - 12\delta_B \ge 0$$
$$\delta_A + \delta_B \le 1$$
$$\delta_A, \delta_B \in \{0,1\}$$

B)
$$x_2 \le 4$$
$$x_2 \ge 0$$
$$4x_1 - x_2 \ge 0$$
$$4x_1 + x_2 \le 20$$
$$x_1 + 3\delta_A \le 5$$
$$x_1 - 3\delta_B \ge 0$$
$$\delta_A + \delta_B \ge 1$$
$$\delta_A, \delta_B \in \{0,1\}$$

C)
$$\left\{\begin{array}{l} x_2 \ge 0 \\ x_2 \le 4 \\ 4x_1 - x_2 \ge 0 \\ x_1 \le 2 \end{array}\right\} \cap \left\{\begin{array}{l} x_2 \ge 0 \\ x_2 \le 4 \\ 4x_1 + x_2 \le 20 \\ x_1 \ge 3 \end{array}\right\}$$

D)
$$x_2 \ge 0$$
$$x_2 \le 4$$
$$4x_1 - x_2 \ge 0$$
$$4x_1 + x_2 \le 20$$
$$x_1 \le 2$$
$$x_1 \ge 3$$
$$x_2 \le 2$$

E)
$$\left\{\begin{array}{l} x_2 \ge 0 \\ x_2 \le 4 \\ 4x_1 - x_2 \ge 0 \\ x_1 \le 2 \end{array}\right\} \cup \left\{\begin{array}{l} x_2 \ge 0 \\ x_2 \le 4 \\ 4x_1 + x_2 \le 20 \\ x_1 \ge 3 \end{array}\right\}$$

F)
$$x_2 \ge 0$$
$$x_2 \le 4$$
$$4x_1 - x_2 \ge 0$$
$$4x_1 + x_2 \le 20$$

Cutting the depots

The capacity of a depot used by Tuberg breweries is given by the following LP-model:

$$
\begin{array}{lrcrcrl}
\text{maximize} & x_1 & + & 4x_2 & & & \\
\text{subject to} & x_1 & + & x_2 & \leq & 4 & \quad \text{(a)} \\
 & x_1 & + & 5x_2 & \leq & 15 & \quad \text{(b)} \\
 & x_1 & & & \leq & 3 & \quad \text{(c)} \\
 & x_1, x_2 \geq 0 & & & & &
\end{array}
$$

Q13: Solve the model to LP-optimality. What is the optimal solution?

A) $x_1 = 0, x_2 = 3$ B) $x_1 = 3, x_2 = 1$
C) $x_1 = 3, x_2 = 0$ D) $x_1 = \frac{5}{4}, x_2 = \frac{11}{4}$
E) $x_1 = 0, x_2 = 4$ F) $x_1 = 3, x_2 = \frac{13}{5}$

Q14: What is the corresponding dual solution?

A) $y_1 = \frac{1}{4}, y_2 = \frac{3}{4}, y_3 = 0$ B) $y_1 = 0, y_2 = \frac{3}{4}, y_3 = \frac{1}{4}$
C) $y_1 = \frac{3}{4}, y_2 = 0, y_3 = \frac{1}{4}$ D) $y_1 = 0, y_2 = 0, y_3 = \frac{49}{12}$
E) $y_1 = \frac{49}{16}, y_2 = 0, y_3 = 0$ F) $y_1 = 0, y_2 = \frac{4}{5}, y_3 = \frac{1}{12}$

Q15: The dual problem is to be solved to integer-optimality. The first equation in the simplex tableau is given by

$$y_1 + \frac{5}{4}y_3 - \frac{5}{4}y_4 + \frac{1}{4}y_5 = \frac{1}{4}$$

where $y_4, y_5 \geq 0$ are the slack variables in the dual problem. Derive a Gomory cut from the equation. Which inequality appears (after elimination of the slack variables)?

A) $2y_1 + y_2 \geq \frac{1}{4}$ B) $y_1 + 2y_2 \geq 4$
C) $y_1 - y_2 \geq 4$ D) $4y_1 + y_2 + y_3 \geq 1$
E) $y_1 + 2y_2 + y_3 \geq 2$ F) $y_1 - y_2 \geq 4$

Q16: Lagrangian relax constraint (b) in the primal model using multiplier $\lambda \geq 0$, and solve the corresponding Lagrangian dual. What is the optimal choice of Lagrangian multiplier λ?

A) $\lambda = \frac{1}{4}$ B) $\lambda = \frac{3}{4}$
C) $\lambda = \frac{5}{7}$ D) $\lambda = \frac{2}{3}$
E) $\lambda = 1$ F) $\lambda = \frac{4}{5}$

Cover inequalities

It is well-known that the mathematical symbol “{” (also known as a Tuberg) was invented by the Tuberg breweries long time before it became a standard symbol in mathematics. In a similar way, Cover inequalities were studied by the Tuberg designers more than 100 years ago, in the context of packing most possible beer bottles into a knapsack.

Q17: Consider the inequality

$$8x_1 + 8x_2 + 7x_3 + 5x_4 + 3x_5 + 2x_6 + 2x_7 \leq 11$$

Which of the following inequalities is not a minimal cover inequality?

A) $x_1 + x_2 \leq 1$
B) $x_1 + x_4 \leq 1$
C) $x_2 + x_5 + x_6 \leq 2$
D) $x_3 + x_5 + x_6 \leq 2$
E) $x_3 + x_4 + x_7 \leq 2$
F) $x_2 + x_4 \leq 1$

Q18: A valid inequality is

$$x_4 + x_5 + x_6 + x_7 \leq 3$$

What is the largest value of α such that the inequality

$$\alpha x_1 + x_4 + x_5 + x_6 + x_7 \leq 3$$

is valid?

A) $\alpha = -1$
B) $\alpha = 0$
C) $\alpha = 1$
D) $\alpha = 2$
E) $\alpha = 3$
F) $\alpha = 4$

In general we may let $N = \{1, \ldots, n\}$ and consider the knapsack polytope P given by

$$P = \text{conv}\left\{ x \in \mathbb{B}^n : \sum_{j \in N} a_j x_j \leq b \right\}$$

Assume that all $a_j \geq 0, j = 1, \ldots, n$ and $b \geq 0$.

Q19: **(text question)** Prove that the dimension of P is

$$\dim(P) = n - |B|$$

where $|B| = \{j \in N : a_j > b\}$.

Q20: **(text question)** Assume that $a_j \leq b$ for $j \in N$. Prove that for a given $j \in N$ the inequality

$$x_j \leq 1$$

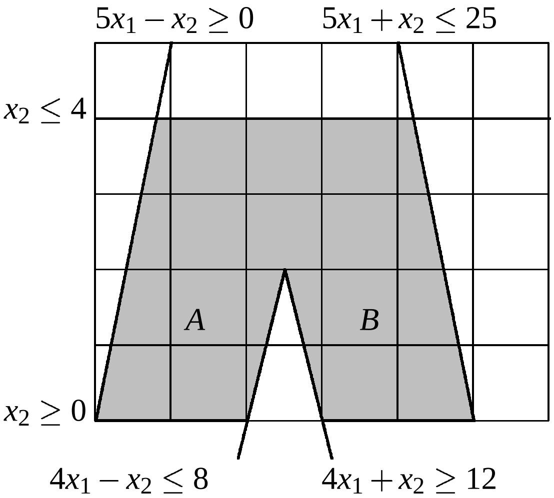
is facet defining for P if and only if

$$a_j + \left(\max_{i \in N \setminus \{j\}} a_i \right) \leq b$$

THE END

Answers

Q11: Since the set T is not convex, we need to introduce some indicator variables.



Let $\delta_A = 1$ if $(x_1, x_2) \in A$ and $\delta_B = 1$ if $(x_1, x_2) \in B$. Then we have

$$\delta_A = 1 \Rightarrow 4x_1 - x_2 \leq 8$$

which can be modeled by the inequality

$$4x_1 - x_2 + M_A \delta_A \leq M_A + 8$$

where M_A is an upper bound on $4x_1 - x_2 - 8$. Since $0 \leq x_1 \leq 5$ and $0 \leq x_2 \leq 4$ we may choose $M_A = 12$, getting the inequality

$$4x_1 - x_2 + 12\delta_A \leq 20$$

In a similar way we have

$$\delta_B = 1 \Rightarrow 4x_1 + x_2 \geq 12$$

which can be modeled by the inequality

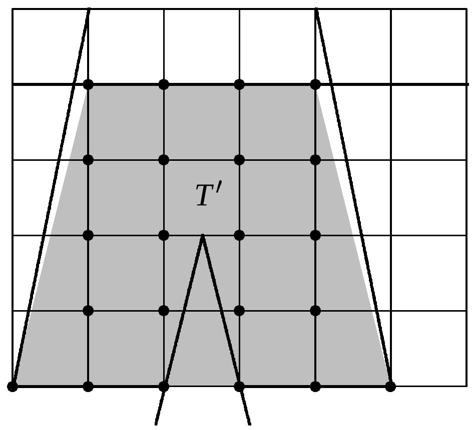
$$4x_1 + x_2 + m_B \delta_B \geq m_B + 12$$

As a lower bound on $4x_1 + x_2 - 12$ we may choose $m_B = -12$, getting the inequality

$$4x_1 + x_2 - 12\delta_B \geq 0$$

Finally we must demand that $\delta_A + \delta_B \geq 1$ to ensure that (x_1, x_2) is in at least one of the sets A or B. Hence the correct answer is 11B).

Q12: The set T' is graphically depicted as

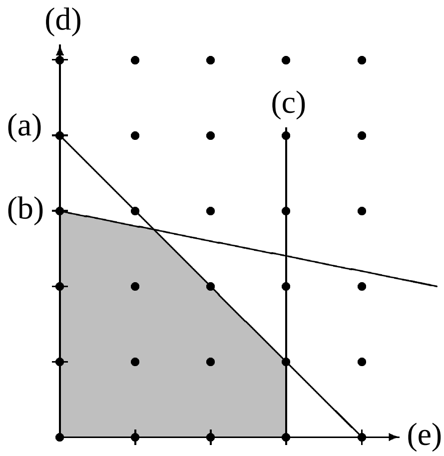


Fortunately, the set T' is convex, hence it is straightforward to describe as an LP-model:

$$\begin{aligned} &x_2 \geq 0 \\ &x_2 \leq 4 \\ &4x_1 - x_2 \geq 0 \\ &4x_1 + x_2 \leq 20 \end{aligned}$$

Hence, the correct answer is 12F).

Q13: Since the problem is defined in two variables we may solve it graphically



It is easily seen that $x_1 = \frac{5}{4}$, $x_2 = \frac{11}{4}$ is the optimal solution, hence 13D) is the correct answer.

Q14: The dual problem is

$$\begin{array}{lccccccc} \text{minimize} & 4y_1 & + & 15y_2 & + & 3y_3 & & \\ \text{subject to} & y_1 & + & y_2 & + & y_3 & \geq & 1 \\ & y_1 & + & 5y_2 & & & \geq & 4 \\ & y_1 & , & y_2 & , & y_3 & \geq & 0 \end{array}$$

From complementary slackness we have

$$\begin{array}{ll} x_1 + x_2 = 4 & \text{hence } y_1 \geq 0 \\ x_1 + 5x_2 = 15 & \text{hence } y_2 \geq 0 \\ x_1 < 3 & \text{hence } y_3 = 0 \end{array}$$

Inserting $y_3 = 0$ in the dual problem we get

$$\begin{array}{ll} x_1 > 0 & \text{hence } y_1 + y_2 = 1 \\ x_2 > 0 & \text{hence } y_1 + 5y_2 = 4 \end{array}$$

with optimal solution $y_1 = \frac{1}{4}$ and $y_2 = \frac{3}{4}$. Hence the correct table must be 14A).

Q15: We have the inequality

$$y_1 + \frac{5}{4}y_3 - \frac{5}{4}y_4 + \frac{1}{4}y_5 = \frac{1}{4}$$

The Gomory cut becomes

$$\frac{1}{4}y_3 + \frac{3}{4}y_4 + \frac{1}{4}y_5 \geq \frac{1}{4}$$

Introducing the slack variables $y_4 = y_1 + y_2 + y_3 - 1$ and $y_5 = y_1 + 5y_2 - 4$ we get

$$\frac{1}{4}y_3 + \frac{3}{4}(y_1 + y_2 + y_3 - 1) + \frac{1}{4}(y_1 + 5y_2 - 4) \geq \frac{1}{4}$$

which is reduced to

$$y_1 + 2y_2 + y_3 \geq 2$$

The correct answer is 15E).

Q16: The optimal choice of Lagrangian multiplier corresponds to the associated dual variable y_2. Hence $\lambda = y_2 = \frac{3}{4}$, and the correct answer is 16B).

Q17: We have the inequality

$$8x_1 + 8x_2 + 7x_3 + 5x_4 + 3x_5 + 2x_6 + 2x_7 \leq 11$$

The cover inequality

$$x_3 + x_4 + x_7 \leq 2$$

is not minimal since if we removed x_7 from the cover we would get the sum $7 + 5 > 11$. Hence the cover is not minimal and 17E) is correct.

Q18: We have the minimal cover inequality

$$x_4 + x_5 + x_6 + x_7 \leq 3$$

To find the largest value of α such that the inequality

$$\alpha x_1 + x_4 + x_5 + x_6 + x_7 \leq 3$$

is valid we solve the problem:

$$\begin{aligned} \gamma = \quad & \text{maximize} && x_4 + x_5 + x_6 + x_7 \\ & \text{subject to} && 8 + 5x_4 + 3x_5 + 2x_6 + 2x_7 \leq 11 \\ & && x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \end{aligned}$$

with optimal solution $\gamma = 1$. Hence $\alpha = 3 - \gamma = 2$, and thus 18D) is correct.

Q19: We first show that $d \geq n - |B|$. Construct the $n - |B|$ points

$$\mathbf{x}^j = (x_1, \ldots, x_n) = (0, \ldots, \underset{\substack{j \\ \downarrow}}{1}, \ldots, 0)$$

for $j \in N \setminus B$. Moreover, construct the additional point $(0, \ldots, 0)$. It is easily checked that the $n - |B| + 1$ points are affinely independent and that all the points are in P. Hence $d \geq n - |B|$.

We next show that $d \leq n - |B|$. This is obvious since any feasible solution must have $x_j = 0$ for all $j \in B$.

Q20: Due to the assumption $a_j \leq b$ for $j \in N$ we know from Q19 that the dimension of P is $\dim(P) = n$. To show that the inequality

$$x_j \leq 1$$

is facet defining we consider the face $F = \{x \in P : x_j = 1\}$. Construct n affine independent points as follows:

$$\mathbf{x}^j = (x_1, \ldots, x_n) = (0, \ldots, 0, \underset{\substack{j \\ \downarrow}}{1}, 0 \ldots, 0)$$

$$\mathbf{x}^{ij} = (x_1, \ldots, x_n) = (0, \ldots, 0, \underset{\substack{i \\ \downarrow}}{1}, 0, \ldots, 0, \underset{\substack{j \\ \downarrow}}{1}, 0, \ldots, 0) \qquad i \in N \setminus \{j\}$$

All the points are in F since $x_j = 1$ and $a_i + a_j \leq b$. Hence $\dim(F) = n - 1$ and thus the inequality $x_j \leq 1$ is facet defining.

Reversely, assume that $x_j \leq 1$ is a facet defining inequality for P. Then the face $F = \{x \in P : x_j = 1\}$ contains n affine independant points $\mathbf{x}^1, \ldots, \mathbf{x}^n$. Since the points are in F we must have $\mathbf{x}^i_j = 1$. Construct an $n \times n$ matrix A whose rows are the coordinates of $\mathbf{x}^i$. If

$$a_j + \left(\max_{i \in N \setminus \{j\}} a_i \right) \leq b \qquad (*)$$

does not hold, then let $k = \arg\max_{i \in N \setminus \{j\}} a_i$. We must have $\mathbf{x}^i_k = 0$ for all $i = 0, \ldots, n$ since $a_j + a_k > b$. Hence matrix A is sigular, and the points $\mathbf{x}^1, \ldots, \mathbf{x}^n$ cannot be affine independant. Thus (*) must hold.