

Introduction to Optimization:

Written Exam, 16 December 1999

Your assignment

20 different questions Q1-Q20 are posed on the subsequent pages. Q1-Q8 and Q11-Q18 are *multiple choice questions*. For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation to write, for example, "7.b" as your answer to question Q7. Q9-Q10 and Q19-Q20 are ordinary *text questions*. Each correct answer to a multiple choice question gives 4 points, to a text question gives 9 points. The maximum score is thus 100 points.

Note: only the last 10 questions are available

Planning the study

A DIKU-student is working in company XYZ to make some money for his studying. During the low-activity hours between 14 and 16, he is planning the courses for the next semester. Looking in the study-plan (*lektionskataloget*) he finds five different courses A, B, C, D and E which seem to be interesting, since they deal with algorithms and combinatorial optimization. However, there are some complicating constraints as follows:

- Courses A and B are both at Friday morning, and thus cannot be followed at the same time.
- To follow courses B or C it is necessary also to follow course D.
- Obviously he cannot follow a course without having the corresponding text book. The cost of books for the individual courses is given in the table below, and the student cannot afford books for more than 1100 kr.
- Some courses are easier than others. Having spoken with some older students, he finds out, that the minimum number of weekly hours needed to pass the exam are as listed in the following table.

course	A	B	C	D	E
book price	300 kr	200 kr	800 kr	400 kr	450 kr
hours preparation	10 h	15 h	13 h	8 h	12 h

- The courses A,B,C are quite mathematically oriented, while the courses D,E are practically oriented. If the student is using more than 25 hours a week on the mathematical courses, he would like also to use 10 hours on the practical courses.
- The student wishes to follow in a satisfactory way most possible courses.

Let $\delta_i \in \{0, 1\}$ indicate whether course i is taken, and $x_i \geq 0$ be the corresponding number of hours used for preparation. The objective function is obviously

$$\text{maximize } \delta_A + \delta_B + \delta_C + \delta_D + \delta_E$$

Q 11 How would you ensure that $\delta_i = 1$ only if the student follows the course using the appropriate number of hours for preparation

$$11.a) \delta_A \geq x_A, \quad \delta_B \geq x_B, \quad \delta_C \geq x_C, \quad \delta_D \geq x_D, \quad \delta_E \geq x_E.$$

$$11.b) \quad 10\delta_A \leq x_A, \quad 15\delta_B \leq x_B, \quad 13\delta_C \leq x_C, \quad 8\delta_D \leq x_D, \quad 12\delta_E \leq x_E.$$

$$11.c) \quad \delta_A \leq 10x_A, \quad \delta_B \leq 15x_B, \quad \delta_C \leq 13x_C, \quad \delta_D \leq 8x_D, \quad \delta_E \leq 12x_E.$$

$$11.d) \quad x_A - 80\delta_A \geq 0, \quad x_B - 80\delta_B \geq 0, \quad x_C - 80\delta_C \geq 0, \quad x_D - 80\delta_D \geq 0, \\ x_E - 80\delta_E \geq 0.$$

$$11.e) \quad x_A - 10\delta_A \geq 0, \quad x_B - 15\delta_B \geq 0, \quad x_C - 13\delta_C \geq 0, \quad x_D - 8\delta_D \geq 0, \\ x_E - 12\delta_E \geq 0.$$

□

To finish the model, the student writes down the following constraints

$$30\delta_A + 20\delta_B + 80\delta_C + 40\delta_D + 45\delta_E \leq 110 \quad (1)$$

$$x_D + x_E \geq \frac{10}{25}(x_A + x_B + x_C) \quad (2)$$

$$\delta_B \leq \delta_D \quad (3)$$

$$\delta_C \leq \delta_D \quad (4)$$

$$x_A + x_B + x_C - 144\delta \leq 24 \quad (5)$$

$$x_D + x_E - 10\delta \geq 0 \quad (6)$$

$$\delta_A + \delta_B \leq 1 \quad (7)$$

$$\delta_A, \delta_B, \delta_C, \delta_D, \delta_E, \delta \in \{0, 1\} \quad (8)$$

$$x_A, x_B, x_C, x_D, x_E \geq 0 \quad (9)$$

Q 12 Unfortunately his model is not correct. Which inequality or inequalities should be removed to get a proper formulation of his problem?

12.a) Inequality (1) should be removed.

12.b) Inequalities (5) and (6) should be removed.

12.c) Inequality (7) should be removed.

12.d) Inequalities (3) and (4) should be removed.

12.e) Inequality (2) should be removed.

12.f) Inequalities (8) and (9) should be removed.

□

Solving the (now correct) model to LP-optimality gave a fractional solution

$$\delta_A = 0, \quad \delta_B = 1, \quad \delta_C = \frac{1}{16}, \quad \delta_D = 1, \quad \delta_E = 1$$

To tighten the formulation, the student wants to derive a cover inequality from the constraint

$$30\delta_A + 20\delta_B + 80\delta_C + 40\delta_D + 45\delta_E \leq 110$$

In order to separate the most violated cover inequality he solves a knapsack problem.

Q 13 What is the correct form of this knapsack problem.

13.a)

$$\begin{aligned} \gamma = \min \quad & x'_A + x'_B + x'_C + x'_D + x'_E \\ \text{s.t.} \quad & x'_A + x'_B + x'_C + x'_D + x'_E \geq 3 \\ & x'_A, x'_B, x'_C, x'_D, x'_E \in \{0, 1\} \end{aligned}$$

13.b)

$$\begin{aligned} \gamma = \min \quad & 0x'_A + 1x'_B + \frac{1}{16}x'_C + 1x'_D + 1x'_E \\ \text{s.t.} \quad & 30x'_A + 20x'_B + 80x'_C + 40x'_D + 45x'_E \geq 111 \\ & x'_A, x'_B, x'_C, x'_D, x'_E \in \{0, 1\} \end{aligned}$$

13.c)

$$\begin{aligned} \gamma = \min \quad & 1x'_A + 0x'_B + \frac{15}{16}x'_C + 0x'_D + 0x'_E \\ \text{s.t.} \quad & 30x'_A + 20x'_B + 80x'_C + 40x'_D + 45x'_E \geq 111 \\ & x'_A, x'_B, x'_C, x'_D, x'_E \in \{0, 1\} \end{aligned}$$

13.d)

$$\begin{aligned} \gamma = \min \quad & x'_A + x'_B + x'_C + x'_D + x'_E \\ \text{s.t.} \quad & 30x'_A + 20x'_B + 80x'_C + 40x'_D + 45x'_E \geq 111 \\ & x'_A, x'_B, x'_C, x'_D, x'_E \in \{0, 1\} \end{aligned}$$

13.e)

$$\begin{aligned} \gamma = \max \quad & \delta_A + \delta_B + \delta_C + \delta_D + \delta_E \\ \text{s.t.} \quad & 30\delta_A + 20\delta_B + 80\delta_C + 40\delta_D + 45\delta_E \leq 110 \\ & \delta_A, \delta_B, \delta_C, \delta_D, \delta_E \in \{0, 1\} \end{aligned}$$

□

When solving the knapsack problem in the previous question, there may be several equivalent solutions. In this case, you should choose the solution with most x'_i set to one.

Q 14 What is the derived cover inequality.

14.a) $\delta_B + \delta_C + \delta_D \leq 2$

$$14.b) \delta_A + \delta_B + \delta_C + \delta_D + \delta_E \leq 2$$

$$14.c) \delta_B + \delta_D + \delta_E \leq 3$$

$$14.d) \delta_B + \delta_C + \delta_D + \delta_E \leq 3$$

14.e) No valid cover inequality can be derived.

□

Pasta production

Company XYZ is also producing excellent Italian dishes. For these dishes it is an art to choose the correct type of pasta for a given sauce. There is only have a limited quantity b of the individual pasta types, thus when planning the dishes, a taste-index c is given for a given combination of sauce and pasta, while the matrix A gives the quantity of pasta used for the given dish. This can be formulated as the following linear model

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

where c and x are vectors of length n , b is a vector of length m , and A is an $n \times m$ matrix.

Q 15 Assume that b is a vector of integers. Regardless of the values of b and c , for which one of the following matrices A will all basic solutions be integer valued. All blank entries in the matrices are zero.

$$A_1 = \begin{pmatrix} & 1 & & 1 \\ 1 & & 1 & \\ & & & 1 \\ & 1 & -1 & \\ & & & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & & -1 & 1 \\ & & -1 & -1 \\ & 1 & 1 & \\ & & & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & & & 1 \\ & 1 & -1 & \\ & 1 & 1 & \\ 1 & 1 & -1 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} & 1 & & 1 \\ 1 & & 1 & \\ & 1 & -1 & \\ -1 & & 1 & 1 \end{pmatrix} \quad A_5 = \begin{pmatrix} 1 & & & 1 \\ -1 & -1 & & -1 \\ & 1 & 2 & \\ & & -1 & \\ & & 1 & \end{pmatrix} \quad A_6 = \begin{pmatrix} 5 & & & \\ & 5 & & \\ & & 5 & \\ & & & 5 \end{pmatrix}$$

15.a) Matrix A_1 .

- 15.b) Matrix A_2 .
 15.c) Matrix A_3 .
 15.d) Matrix A_4 .
 15.e) Matrix A_5 .
 15.f) Matrix A_6 .
 15.g) None of the above.

□

Gomory cut

Only a few people know that Gomory actually started his career in Company XYZ. For several years he worked in the kitchen, where he cut out the pizzas. One day after having cut out several hundreds of pizzas he went home and developed *Gomory's cutting plane algorithm*. Gomory's career in company XYZ however ended when he started to arrange the olives at all integer coordinates, and cutting out the pizzas such that no integer points were excluded. The last pizza with olives at integer coordinates corresponded to the following optimization problem:

$$\begin{aligned}
 z = \max \quad & 4x_1 - x_2 \\
 \text{s.t.} \quad & 7x_1 - 2x_2 \leq 14 \\
 & x_2 \leq 3 \\
 & 2x_1 - 2x_2 \leq 3 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

We add slack variable $x_3 \geq 0$ to the first constraint, $x_4 \geq 0$ to the second constraint, and $x_5 \geq 0$ to the last constraint. Running some iterations of the Simplex algorithm, the following equations appear

$$\begin{aligned}
 z &= \frac{59}{7} - \frac{4}{7}x_3 - \frac{1}{7}x_4 \\
 x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 &= \frac{20}{7} \\
 x_2 + x_4 &= 3 \\
 -\frac{2}{7}x_3 + \frac{10}{7}x_4 + x_5 &= \frac{23}{7}
 \end{aligned}$$

Q 16 What is the optimal LP-solution $x = (x_1, x_2, x_3, x_4, x_5)$ to the above problem

16.a) $x = (\frac{20}{7}, 3, 0, 0, \frac{23}{7})$

16.b) $x = (\frac{59}{7}, 0, 0, -\frac{4}{7}, -\frac{1}{7})$

16.c) $x = (0, 0, -\frac{2}{7}, \frac{10}{7}, 1)$

16.d) $x = (\frac{20}{7}, 3, \frac{22}{7}, 0, 0)$

16.e) $x = (-\frac{59}{7}, 0, 0, \frac{4}{7}, \frac{1}{7})$

16.f) $x = (1, 1, 0, 0, 1)$

□

It is now wished to solve the problem to integer optimality. Derive a Gomory cut from the first constraint in the simplex tableau.

Q 17 Which of the following inequalities appears:

17.a) $\frac{1}{7}x_3 + \frac{1}{7}x_4 \geq \frac{6}{7}$

17.b) $x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{20}{7}$

17.c) $\frac{5}{7}x_1 + \frac{1}{7}x_2 \geq 1$

17.d) $\frac{2}{7}x_1 + \frac{1}{7}x_2 + \frac{1}{3}x_3 \geq \frac{1}{3}$

17.e) $x_3 + 2x_4 \geq 6$

□

Modular arithmetics

Professor Wolsey also worked at Company XYZ during one summer, where he was making home-made *fettucine* and *tagliatelle*. Looking at the endless rows of equidistant slices inspired him to develop the theory of modular arithmetics.

In an IP problem we have the constraint

$$21x_1 + 43x_2 + 7x_3 = 35$$

Q 18 Which of the following inequalities can be derived using modular arithmetics

18.a) $x_1 + x_2 + x_3 \geq 7$

$$18.b) \quad 3x_1 + x_2 + x_3 \geq 5$$

$$18.c) \quad 3x_1 + x_2 + x_3 \geq 6$$

$$18.d) \quad 3x_1 + 2x_2 + x_3 \geq 7$$

$$18.e) \quad x_1 + 3x_2 + x_3 \geq 6$$

□

Spaghetti Bolognese (text questions)

The only place in the world, where you cannot order *Spaghetti Bolognese* is actually in Bologna, where this dish is considered to be too simple. Anyhow, to make a good meat sauce for spaghetti Bolognese, you need to mix a quantity of onion x_1 and minced beef x_2 according to the following constraints:

$$\begin{aligned} z = \max \quad & 2x_1 + 7x_2 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 12 \\ & 2x_1 + x_2 \leq 7 \\ & x_2 \geq 1 \\ & x_1 \geq 1 \\ & x_1, x_2 \geq 0, \text{ integer} \end{aligned}$$

Q 19 Relax the first constraint $x_1 + 4x_2 \leq 12$ using a lagrangian multiplier λ . Write down the relaxed problem. □

Q 20 For which value of λ do we obtain the smallest solution value of the lagrangian relaxed problem? □

Planning the study

Answer 11 The correct formulation is

$$10\delta_A \leq x_A, \quad 15\delta_B \leq x_B, \quad 13\delta_C \leq x_C, \quad 8\delta_D \leq x_D, \quad 12\delta_E \leq x_E.$$

since this has the effect that if $\delta_i = 1$ then $x_i \geq m_i$ where m_i is the minimum hours of preparation for course i . Thus 11.b) is the correct answer. ■

Answer 12 Only one of the courses A and B can be followed

$$\delta_A + \delta_B \leq 1$$

That course B or C demands course D is written

$$\begin{aligned} \delta_B &\leq \delta_D \\ \delta_C &\leq \delta_D \end{aligned}$$

To ensure that no more than 1100 kr are used for the books can be expressed as

$$30\delta_A + 20\delta_B + 80\delta_C + 40\delta_D + 45\delta_E \leq 110$$

If the student is using more than 25 hours on courses A,B,C he should also use at least 10 hours on courses D,E. For this purpose we introduce a new boolean variable δ which is 1 when $x_A + x_B + x_C \geq 25$. This leads to the following constraints

$$\begin{aligned} x_A + x_B + x_C - 144\delta &\leq 24 \\ x_D + x_E - 10\delta &\geq 0 \end{aligned}$$

Thus the wrong constraint is

$$x_D + x_E \geq \frac{10}{25}(x_A + x_B + x_C)$$

For instance if $x_A + x_B + x_C = 50$ it will push $x_D + x_E \geq 20$ which was not the intention. Thus the correct answer is 12.e). ■

Answer 13 We use the algorithm from exercise 10. Since we have the solution

$$\delta_A = 0, \quad \delta_B = 1, \quad \delta_C = \frac{1}{16}, \quad \delta_D = 1, \quad \delta_E = 1$$

the correct formulation is

$$\begin{aligned} \gamma = \min \quad & 1x'_A + 0x'_B + \frac{15}{16}x'_C + 0x'_D + 0x'_E \\ \text{s.t.} \quad & 30x'_A + 20x'_B + 80x'_C + 40x'_D + 45x'_E \geq 111 \\ & x'_A, x'_B, x'_C, x'_D, x'_E \in \{0, 1\} \end{aligned}$$

thus 13.c) is correct. ■

Answer 14 The solution to the above knapsack problem is easily found by inspection. Since we should choose the solution with most possible items chosen, we set $x'_B = x'_D = x'_E = 1$, which gives the weight sum 105 in the capacity constraint. To pass 111 we may choose either $x'_A = 1$ or $x'_C = 1$, where $x'_C = 1$ is the choice giving the smallest objective. Thus we have the solution

$$x'_B = x'_C = x'_D = x'_E = 1$$

with objective value $\gamma = \frac{15}{16} < 1$. We may derive the cover inequality

$$\delta_B = \delta_C = \delta_D = \delta_E \leq 4 - 1 = 3$$

and 14.d) is correct.

Solving the problem to LP-optimality with the new constraint added, we find the solution

$$\delta_A = \frac{1}{18}, \delta_B = \frac{17}{18}, \delta_C = \frac{1}{18}, \delta_D = 1, \delta_E = 1$$

with objective value 3.055. Without the new constraint we had the objective value 3.063, thus the constraint did have an effect. The integer-optimal solution is

$$\delta_B = 1, \delta_D = 1, \delta_E = 1, x_B = 15, x_D = 8, x_E = 12.$$

■

Pasta production

Answer 15 If the matrix A is totally unimodular (TU), we will obtain a basic integer solution x to the problem for any choice of c and b as long as b is integral. It is easily seen that A_1 is not TU, since column 2 and 4 contain the sub-matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ which has determinant -2. The same applies for matrix A_2 where column 4 and 5 contain the sub-matrix $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ with determinant 2. Matrix A_3 contains the sub-matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ with determinant 2. None of A_5 and A_6 satisfy the property that all entries are in $\{0,1,-1\}$.

To prove that A_4 is TU, we prove that it satisfies property P (Williams page 208). We divide the columns in two sets: columns $\{1,2,3\}$ and $\{4,5\}$. Every row contains exactly two entries different from zero, and if they have the same sign, they are in different sets. Thus the correct answer is 15.d). ■

Gomory cut

Answer 16 From the objective function it is seen that x_3 and x_4 are not in the basis. The remaining variables are in diagonal form, and all coefficients to nonbasis

variables in the objective function are negative. Thus we know that the algorithm has terminated. The correct solution is read as

$$x_1 = \frac{20}{7}, x_2 = 3, x_5 = \frac{23}{7}$$

Thus answer 16.a) is correct. ■

Answer 17 The first constraint reads

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7}$$

which leads to the Gomory cut

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

which by multiplication with 7 gives 17.e).

Adding the inequality to the problem and solving the model to LP-optimality, we find the solution

$$x_1 = 2, x_2 = \frac{1}{2}, x_3 = 1, x_4 = \frac{5}{2}$$

with objective value 7.5. The original problem had the solution value 8.429 so the cut had a considerable effect. The integer-optimal solution is

$$x_1 = 2, x_2 = 1, x_3 = 2, x_4 = 2, x_5 = 1$$

with objective value 7. ■

Modular arithmetics

Answer 18 We have the equation

$$21x_1 + 43x_2 + 7x_3 = 35$$

taking the remainder modulo 6, we get the inequality

$$3x_1 + x_2 + x_3 \geq 5$$

Thus answer 18.b) is correct. ■

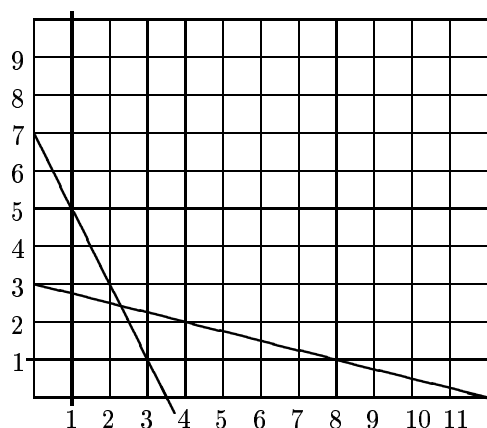
Spaghetti Bolognese

Answer 19 Lagrangian relaxing the first constraint we get the problem

$$\begin{array}{rcll} z = \max & (2 - \lambda)x_1 & + & (7 - 4\lambda)x_2 & + 12\lambda \\ \text{s.t.} & 2x_1 & + & x_2 & \leq 7 \\ & & & x_2 & \geq 1 \\ & x_1 & & & \geq 1 \\ & x_1, x_2 \geq 0, & \text{integer} & & \end{array}$$

■

Answer 20 By drawing the problem we see that the remaining constraints define the convex hull. In this case we know that the best choice of the lagrangian multiplier corresponds to the dual variable of the relaxed constraint.



The LP-optimal solution of the original problem is

$$x_1 = \frac{16}{7}, \quad x_2 = \frac{17}{7}$$

which can be solved graphically. The corresponding objective value is $\frac{151}{7}$. To find the dual variables we use complementary slackness. The dual problem is defined as

$$\begin{aligned} z = \min \quad & 12y_1 + 7y_2 - y_3 - y_4 \\ \text{s.t.} \quad & y_1 + 2y_2 - y_4 \geq 2 \\ & 4y_1 + y_2 - y_3 \geq 7 \\ & x_1, x_2 \geq 0, \text{ integer} \end{aligned}$$

Since both of the primal variables $x_1, x_2 > 0$ the two constraints are binding thus we have

$$\begin{aligned} y_1 + 2y_2 - y_4 &= 2 \\ 4y_1 + y_2 - y_3 &= 7 \end{aligned}$$

We know that $y_3 = y_4 = 0$, since the LP-optimum of the primal problem is not binding for the two last constraints. This melts down to the equations

$$\begin{aligned} y_1 + 2y_2 &= 2 \\ 4y_1 + y_2 &= 7 \end{aligned}$$

with optimal solution

$$y_1 = \frac{12}{7}, \quad y_2 = \frac{1}{7}$$

The optimal choice of λ is $\lambda = y_1 = \frac{12}{7}$. In this case we find that the solution to the lagrangian relaxed problem is $x_1 = 3$, and $x_2 = 1$. The objective then becomes $\frac{151}{7}$, which is equivalent to the LP-solution to the original problem. ■