

Introduction to Optimization:

Written Exam, 17 December 1998

Your assignment

20 different questions Q1-Q20 are posed on the subsequent pages which also provide you with the supplementary information when needed.

For each question a number of possible answers is given. Tick the appropriate box and note that at most one box must be marked for each question since your answer to the question will otherwise be disregarded

Note: only the last 10 questions are available

Questions Q11-Q13: Planning pension

Three professors, Dantzig, Gomory and Chvátal are sitting at the same table during a conference dinner. They enjoy the food very much, and in particular the excellent wine. "I wish that we could afford a good Burgundy every day for the rest of our life", says Chvátal. Professor Dantzig agrees: "Well, if we invest our pension properly, we could actually do this. As a matter of fact, I am using OR models to maximize the profit of my invested pension".

Professor Dantzig can invest his pension in three different projects. The amount of money invested in project i is denoted x_i . In order to avoid paying taxes of his profit, the investment must satisfy the following constraints

$$\begin{array}{llll} \max & x_1 & +x_2 & +x_3 \\ \text{s.t.} & 2x_1 & +4x_2 & +7x_3 \leq 6 \\ & x_1 & -3x_2 & +x_3 \leq 8 \\ & x_1 & +x_2 & -x_3 \leq 1 \\ & x_1, x_2, x_3 & \geq 0 & \end{array}$$

With a little bit of writing on his serviette, Professor Dantzig solves the problem by using the simplex algorithm. But in order to test the other professors, he only announces the dual solution $y_1 = \frac{2}{9}, y_2 = 0, y_3 = \frac{5}{9}$.

Q 11 What is the optimal primal solution of the above LP-problem (variables not listed are assumed to be zero)

11.a) $x_1 = \frac{13}{9}, x_2 = \frac{5}{9}$.

11.b) $x_2 = \frac{17}{9}$.

11.c) $x_1 = x_2 = x_3 = 1$.

11.d) $x_1 = \frac{13}{9}, x_3 = \frac{4}{9}$.

11.e) $x_3 = \frac{13}{9}$.

□

Professor Chvátal has been drinking too much of the good Burgundy and his nose is starting to shine in a pink red color. Looking at the solution he says: "This is not good enough. All our variables must attain integer values".

The other professors agree, but unfortunately this makes the problem considerably more difficult to solve. Professor Gomory however proposes an inequality which might make the problem easier to solve.

Q 12 One of the lines in the simplex tableau looks as:

$$x_1 = \frac{13}{9} - \frac{11}{9}x_2 - \frac{1}{9}s_1 - \frac{7}{9}s_3$$

where s_i is the slack variable of constraint i . Derive a Gomory cut from the above equation, using the formula from Nemhauser and Wolsey. Which one of the following inequalities appears?

12.a) $18x_1 + 27x_2 + 54x_3 \leq 43$

12.b) $x_1 + x_2 \leq 1$

12.c) $\frac{1}{9}x_1 + \frac{8}{9}x_3 \geq 3$

12.d) $\frac{1}{3}x_2 + \frac{3}{4}x_3 \leq \frac{1}{7}$

12.e) $x_1 + 2x_2 + x_3 \leq 5$

12.f) $\frac{4}{9} + \frac{7}{9}x_1 + \frac{8}{9}x_2 + \frac{2}{9}x_3 \geq 1$

□

In the meantime, professor Chvátal has found the valid inequality

$$2x_1 + 2x_2 + 3x_3 \leq 5$$

Q 13 How can this inequality be obtained.

13.a) The inequality is a Chvátal-cut obtained by using multipliers (2, 2, 1).

13.b) The inequality is a Chvátal-cut obtained by using multipliers $(\frac{1}{2}, \frac{3}{2}, \frac{1}{2})$.13.c) The inequality is a Chvátal-cut obtained by using multipliers $(0, \frac{1}{2}, \frac{2}{3})$.

13.d) The inequality is a Chvátal-cut obtained by using multipliers (2, 1, 3).

13.e) The inequality is a Chvátal-cut obtained by using multipliers (1, 1, 2).

13.f) The inequality is not valid since it excludes integer feasible solutions.

□

Questions Q14-Q16: Still active

As professors get older, they get more concerned about which projects they should become engaged in. Professor P7 is planning his next semester, where he would like to work on five different projects 1, 2, 3, 4, 5. Every week, he has at most 37 hours available for working on the projects. If he chooses to work on any project i , then he must work at least 5 hours on that project every week. Project 2 is in cooperation with professor X and project 3 involves professor Y . Since the two professors X and Y cannot stand each other, it will not be possible to work on both project 2 and 3. Project 1, however, involves some theorems, which will be developed as part of projects 3 and 4. Thus if he chooses to work on project 1, there must be at least 20 hours of work every week on projects 3 and 4. On the other hand, if at least 25 hours a week are used in total on projects 1, 3, 4 and 5, then at least 5 hours should be used on project 2.

Using variables x_i to mean the number of hours used on project i , Professor P7 introduces indicator variables δ_i to indicate whether he should get involved in project i .

Professor P7 would like to maximize the number of projects he is involved in. Thus the natural objective value is

$$\text{maximize } \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5$$

Q 14 Which of the following constraints should be *removed* to get a proper formulation of the problem. (You may assume that x_i is measured in whole hours, i.e. $\epsilon = 1$ in the equations Williams presents in Chapter 9).

14.a) $x_1 + x_2 + x_3 + x_4 + x_5 \leq 37$

14.b) $x_i - 5\delta_i \geq 0$, for $i = 1, \dots, 5$

14.c) $x_i - 37\delta_i \leq 0$, for $i = 1, \dots, 5$

14.d) $\delta_2 + \delta_3 \leq 1$

14.e) $x_3 + x_4 - 20\delta_1 \geq 0$

14.f) $x_1 + x_3 + x_4 + x_5 + 26\delta_2 \geq 26$

14.g) $x_1 + x_3 + x_4 + x_5 - 13\delta_2 \leq 24$

14.h) $\delta_i \in \{0, 1\}$, for $i = 1, \dots, 5$

□

Time is not the only resource for finishing a project. Also money is needed to cover travel expenses, conferences, and computing facilities. Professor P7 has a budget of 20 (thousand kroner) to cover such expenses. Knowing the money needed for each project, he derives the following inequality:

$$5\delta_1 + 8\delta_2 + 4\delta_3 + 9\delta_4 + 6\delta_5 \leq 20$$

Q 15 Some of the indices form a minimal cover (see Williams chapter 10 for a definition). Which set of indices does *not* form a minimal cover. We assume that a_i is the coefficient of δ_i , i.e. $a_1 = 5$, $a_2 = 8$, $a_3 = 4$, $a_4 = 9$, and $a_5 = 6$.

15.a) indices $\{1, 2, 4\}$.

15.b) indices $\{1, 2, 3, 5\}$.

15.c) indices $\{1, 4, 5\}$.

15.d) indices $\{2, 3, 4\}$.

15.e) indices $\{2, 4, 5\}$.

15.f) all the proposed sets form a minimal cover.

□

Q 16 One of the minimal covers can be extended to a strong cover inequality of the form

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 \leq m$$

What is the smallest, correct value of m

16.a) $m = 1$

16.b) $m = 2$

16.c) $m = 3$

16.d) $m = 4$

16.e) $m = 5$

□

Questions Q17-Q20: Turing Medal

An academic career is not complete without having obtained some medals or awards. Thus professor P7 is working hard on getting the *Turing Medal*, which is the equivalent of a Nobel Prize in Combinatorial Optimization. By solving the following very hard problem, professor P7 is almost sure of getting the *Turing Medal* before year 2000.

$$\begin{array}{rcllcl}
 \max & x_1 & +x_2 & +x_3 & -x_4 & -x_5 & & \\
 \text{s.t.} & x_1 & -x_2 & & & & \leq & 3 \\
 & & x_2 & & & -x_5 & \leq & 6 \\
 & x_1 & +x_2 & +x_3 & & & \leq & 8 \\
 & x_1 & & -x_3 & & & \leq & 7 \\
 & & x_2 & & -x_4 & & \leq & 17 \\
 & x_i & \geq 0, & i = 1, \dots, 5 & & & &
 \end{array}$$

The simplex algorithm returns the solution $x_1 = \frac{11}{2}$, $x_2 = \frac{5}{2}$, $x_3 = x_4 = x_5 = 0$. Use complementary slackness to find the dual solution.

Q 17 What is the value of the first dual variable y_1 ?

17.a) $y_1 = 0$

17.b) $y_1 = \frac{11}{2}$

17.c) $y_1 = 4$

17.d) $y_1 = 1$

17.e) $y_1 = \frac{3}{2}$

17.f) $y_1 = 2$

17.g) $y_1 = -1$

□

Professor P7 would like to solve the problem to integer optimality but, as mentioned above, the simplex algorithm returns a solution which is not particularly integral. Thinking about the problem for a couple of days, Professor P7 gets a bright idea: If we remove one of the constraints, the remaining constraint matrix satisfies property P (Williams, Chapter 10) and is thus totally unimodular.

Q 18 For the constraint matrix to satisfy property P, which constraint should be removed?

18.a) $x_1 - x_2 \leq 3$

18.b) $x_2 - x_5 \leq 6$

18.c) $x_1 + x_2 + x_3 \leq 8$

18.d) $x_1 - x_3 \leq 7$

18.e) $x_2 - x_4 \leq 17$

□

Instead of removing the constraint, one can use Lagrangian relaxation to somehow punish violated constraints. Using multiplier $\lambda = 2$ for the constraint answering Q18, a reduced problem with only four constraints results.

Q 19 What is the objective function of the reduced problem

19.a) $-x_1 + 3x_2 + x_3 - x_4 - x_5 + 6$

19.b) $x_1 - x_2 + x_3 - x_4 + x_5 + 12$

19.c) $3x_1 + 3x_2 + 3x_3 - x_4 - x_5 - 16$

19.d) $-x_1 - x_2 - x_3 - x_4 - x_5 + 16$

19.e) $3x_1 + x_2 - x_3 - x_4 - x_5 - 14$

19.f) $x_1 - x_2 + x_3 - 3x_4 - x_5$

19.g) $x_1 - x_2 + x_3 - 3x_4 - x_5 + 34$

□

Since the constraint matrix of the reduced problem is Totally Unimodular, the multiplier λ leading to the tightest upper bound may be found through linear programming.

Q 20 What is the choice of λ leading to the tightest upper bound

20.a) $\lambda = -3$.

20.b) $\lambda = 1$.

20.c) $\lambda = \frac{11}{2}$.

20.d) $\lambda = 7$.

20.e) $\lambda = -1$.

20.f) $\lambda = \frac{5}{2}$.

20.g) $\lambda = 0$.

□

Questions Q11-Q13: Planning pension

Answer 11 The optimal solution can be found from the dual solution through complementary slackness. In our case, we have five proposals given, and thus it is much easier to verify a solution. The optimal solution must satisfy:

- The primal solution should be feasible.
- The maximum objective value $x_1 + x_2 + x_3$ equals the dual solution value $6y_1 + 8y_2 + y_3 = 6 \cdot \frac{2}{9} + 8 \cdot 0 + \frac{5}{9} = \frac{17}{9}$.

We have: 11.a) does not obtain the correct objective value, 11.b) has the correct objective value, but it is not feasible (last constraint is violated), 11.c) has a too large objective value, 11.d) has the correct objective value and is feasible, 11.e) does not obtain the correct objective value. Thus answer 11.d) is the correct one. ■

Answer 12 Using the formula from Nemhauser and Wolsey page 212, we have the equation

$$x_1 + \frac{11}{9}x_2 + \frac{1}{9}s_1 + \frac{7}{9}s_3 = \frac{13}{9}$$

which gives the Gomory cutting plane

$$\frac{2}{9}x_2 + \frac{1}{9}s_1 + \frac{7}{9}s_3 \geq \frac{4}{9}$$

Substituting the proper values of s_1 and s_2 one gets

$$x_1 + x_2 \leq 1$$

Thus 12.b) is correct. Adding this constraint to the model, one gets the objective 1.57143 which is considerably tighter than the original LP-solution of 1.88889. ■

Answer 13 Using multipliers (2, 1, 3) for the three constraints and adding them together, one gets:

$$8x_1 + 8x_2 + 12x_3 \leq 23$$

dividing by four and rounding down, one gets

$$2x_1 + 2x_2 + 3x_3 \leq \left\lfloor \frac{23}{4} \right\rfloor = 5$$

Thus 13.d) is correct. Adding this constraint to the model as well as the previous Gomory cuts, one gets the objective 1.57143, which did not improve the objective. ■

Questions Q14-Q16: Still active

Answer 14 The proper formulation is: The professor may work at most 37 hours a week:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 37$$

We introduce binary decision variables δ_i which attain the value $\delta_i = 1$ iff the professor becomes engaged in project i . Since the professor will work at least 5 hours, if he starts working on a project, we may choose $\epsilon = 5$. An upper bound M on x_i is 37.

$$x_i - 5\delta_i \geq 0 \text{ for } i = 1, \dots, 5$$

$$x_i - 37\delta_i \leq 0 \text{ for } i = 1, \dots, 5$$

Only one of the projects 2 and 3 can be started:

$$\delta_2 + \delta_3 \leq 1$$

If project 1 is started, then at least 20 hours should be used on projects 3 and 4. More formally this means $\delta_1 = 1 \Rightarrow x_3 + x_4 \geq 20$. A lower bound m on $x_3 + x_4 - 20$ is $m = -20$, thus we get the formulation

$$x_3 + x_4 - 20\delta_1 \geq 0$$

If at least 25 hours a week are used on projects 1, 3, 4, 5 then at least 5 hours should be used on project 2. Since the minimal working effort on a project is 5 hours, this is equivalent to saying $x_1 + x_3 + x_4 + x_5 \geq 25 \Rightarrow \delta_2 = 1$. An upper bound M on $x_1 + x_3 + x_4 + x_5 - 25$ is $37 - 25 = 12$. As stated in the question we may set $\epsilon = 1$ thus getting

$$x_1 + x_3 + x_4 + x_5 - 13\delta_2 \leq 24$$

All δ_i are binary variables:

$$\delta_i \in \{0, 1\} \text{ for } i = 1, \dots, 5$$

Now to the *wrong* part: The inequality 14.f) has the form

$$x_1 + x_3 + x_4 + x_5 - 26\delta_2 \geq 26$$

which says $x_1 + x_3 + x_4 + x_5 \leq 25 \Rightarrow \delta_2 = 1$. This is exactly the reverse of what the professor demanded, so 14.f) is wrong. ■

Answer 15 If we denote the indices by a_i we have an inequality of the form

$$a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_4 + a_5\delta_5 \leq 20$$

In 15.c) the sum of $a_3 + a_4 + a_5 = 4 + 9 + 6$ does not exceed 20 and is accordingly not a cover. In particular, it is not a *minimal* cover. ■

Answer 16 The indices $\{1, 2, 3, 5\}$ can be extended to a cover $\{1, 2, 3, 4, 5\}$ since $a_4 \geq a_i$ for $i = 1, 2, 3, 5$. In this way we get the cover inequality

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 \leq 3$$

Thus the correct answer is $m = 3$ as stated in 16.c). ■

Questions Q17-Q20: Touring Medal

Answer 17 One could simply test which of the given solutions is a correct dual solution. But the correct way of solving the problem is to use complementary slackness.

The first inequality is tight since $x_1 - x_2 = 3$ so the dual variable y_1 may attain a nonnegative value. The third inequality is also tight, since $x_1 + x_2 + x_3 = 8$, so the same applies for y_3 . All dual variables, corresponding to constraints which are not binding, have the value zero, i.e. $y_2 = y_4 = y_5 = 0$. Using the arguments in the reverse form for the primal variables, we know that $x_1 \neq 0$ and $x_2 \neq 0$. Looking at the dual constraints, one must have $y_1 + y_3 + y_4 = 1$ and $-y_1 + y_2 + y_3 + y_5 = 1$. Since $y_2 = y_4 = y_5 = 0$ we can immediately see that $y_1 = 0$ and $y_3 = 1$. Thus the correct answer is 17.a). ■

Answer 18 If we remove the third constraint $x_1 + x_2 + x_3 \leq 8$ then the matrix corresponding to the left-hand side of the problem, satisfies property P: Every row has at most two elements. All elements are 0, 1, -1. Moreover, if we assign all columns to one class, then we satisfy the last criteria: If a row contains two non-zero elements of different sign, then they belong to the same class. Thus the correct answer is 18.c). ■

Answer 19 Lagrangian relaxing the constraint $x_1 + x_2 + x_3 \leq 8$ we get the objective function

$$x_1 + x_2 + x_3 - x_4 - x_5 - 2(x_1 + x_2 + x_3 - 8)$$

which gives

$$-x_1 - x_2 - x_3 - x_4 - x_5 + 16$$

Thus the correct answer is 19.d). ■

Answer 20 If the remaining problem is totally unimodular after having Lagrangian relaxed some constraints, then the best choice of λ is to choose the dual variables associated with the relaxed constraints. From the previous questions we know that $y_3 = 1$ thus choosing $\lambda = 1$ leads to the tightest bound, and 20.b) is correct. ■