

Introduction to Optimization:

Written Exam, 14 December 2001

Your assignment

20 different questions Q1-Q20 are posed on the subsequent pages. Q1-Q8 and Q11-Q18 are *multiple choice questions*. For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation to write, for example, "7.b" as your answer to question Q7. Q9-Q10 and Q19-Q20 are ordinary *text questions*. Each correct answer to a multiple choice question gives 4 points, to a text question gives 9 points. The maximum score is thus 100 points.

Note: only the last 10 questions are available

Q 14: What is the optimal solution value of the problem

14.a) $z = \frac{92}{3}$

14.d) $z = 38\frac{3}{7}$

14.b) $z = \frac{78}{5}$

14.e) $z = 60$

14.c) $z = \frac{1}{45}$

14.f) $z = \frac{157}{2}$

Gomory cut

The following optimization problem is given

$$\begin{aligned} & \text{maximize} && 60x_1 + 40x_2 + 20x_3 \\ & \text{subject to} && 8x_1 + 6x_2 + x_3 \leq 48 \\ & && 8x_1 + 4x_2 + 3x_3 \leq 40 \\ & && 4x_1 + 3x_2 + x_3 \leq 16 \\ & && x_1, x_2, x_3 \geq 0, \text{ integer} \end{aligned}$$

Having added slack variables s_1, s_2, s_3 to the three constraints, and solving the LP-relaxed problem with the Simplex algorithm the following tableau appears

$$\begin{array}{rcccccccl} z & + & 4x_1 & & & + & 8s_2 & + & 16s_3 & = & 288 \\ & & 1.6x_1 & & & + & s_1 & + & 1.2s_2 & - & 5.6s_3 & = & 27.2 \\ & & 1.6x_1 & & + & x_3 & & + & 1.2s_2 & - & 1.6s_3 & = & 11.2 \\ & & 0.8x_1 & + & x_2 & & & - & 0.4s_2 & + & 1.2s_3 & = & 1.6 \end{array}$$

Derive a Gomory cut from the first inequality having a fractional solution variable.

Q 15: Which inequality appears:

15.a) $\frac{1}{5}x_1 + \frac{2}{5}s_2 + \frac{3}{5}s_3 \geq \frac{1}{5}$

15.d) $s_1 \geq \frac{1}{5}$

15.b) $\frac{3}{5}x_1 + s_1 + \frac{1}{5}s_2 + \frac{2}{5}s_3 \geq 27$

15.e) $\frac{3}{5}x_1 - s_1 + \frac{1}{5}s_2 + \frac{3}{5}s_3 \geq 27.2$

15.c) $\frac{3}{5}x_1 + \frac{1}{5}s_2 + \frac{2}{5}s_3 \geq \frac{1}{5}$

15.f) $\frac{3}{5}x_1 + \frac{1}{5}s_2 - \frac{3}{5}s_3 \geq \frac{1}{5}$

Playing Bridge

Playing bridge is relaxing occupation to most people but professors P_1 to P_8 are very serious players: Before playing each card an LP-model is formulated to maximize the probability of getting the next trick. A model \mathcal{A} is given by

$$\begin{aligned} & \text{maximize} && x_1 + 4x_2 \\ & \text{subject to} && x_1 + x_2 \leq 5 && \text{(a)} \\ & && x_1 + 5x_2 \leq 20 && \text{(b)} \\ & && 5x_1 + 3x_2 \leq 20 && \text{(c)} \\ & && x_1 && \leq 3 && \text{(d)} \\ & && x_1 && \geq 0 && \text{(e)} \\ & && && x_2 && \geq 0 && \text{(f)} \\ & && x_1, x_2 && \text{integer} \end{aligned}$$

Q 16: Which of the following statements is correct

- 16.a) inequalities (a), (e), (f) are facet defining and inequality (d) is redundant.
 16.b) inequalities (d), (e), (f) are facet defining and inequality (a) is redundant.
 16.c) inequalities (a), (d), (e), (f) are facet defining.
 16.d) all the inequalities (a), (b), (c), (d), (e), (f) are facet defining.
 16.e) inequalities (e), (f) are facet defining and inequalities (b), (c) are redundant.
 16.f) inequalities (c), (d), (e), (f) are redundant.

Q 17: Consider the problem \mathcal{A} . Derive the facet defining inequality

$$x_1 + 2x_2 \leq 8$$

as a Chvatal-Gomory cut using the multipliers $u = (u_a, u_b, u_c, u_d, u_e, u_f)$ for the six inequalities (a) to (f). What is the correct choice of multipliers:

- 17.a) $u = (\frac{1}{4}, \frac{3}{4}, 0, 0, 2, 1)$ 17.d) $u = (2, 3, 1, 0, 0, 0)$
 17.b) $u = (3, 4, 0, 0, -1, -1)$ 17.e) $u = (0, \frac{1}{5}, \frac{3}{5}, 0, -1, 0)$
 17.c) $u = (1, 0, \frac{1}{2}, \frac{3}{4}, 0, -1)$ 17.f) $u = (\frac{3}{4}, \frac{1}{4}, 0, 0, 0, 0)$

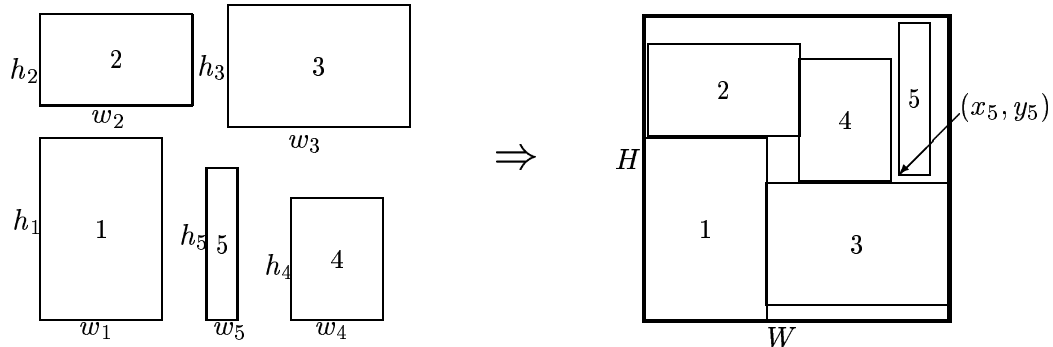
Q 18: Lagrangian relax constraint (b) in problem \mathcal{A} using a multiplier $\lambda \geq 0$. What is the resulting problem?

- | | | | |
|-------|---|-------|--|
| 18.a) | $\begin{aligned} & \max(1 - \lambda)x_1 + (4 - 5\lambda)x_2 + 20\lambda \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 5x_1 + 3x_2 \leq 20 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \text{ integer} \end{aligned}$ | 18.d) | $\begin{aligned} & \max(1 - \lambda)x_1 + (4 - 5\lambda)x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 5x_1 + 3x_2 \leq 20 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \text{ integer} \end{aligned}$ |
| 18.b) | $\begin{aligned} & \max x_1 + x_2 \\ \text{s.t.} \quad & (1 + \lambda)x_1 + (1 + 5\lambda)x_2 \leq 5 + 20\lambda \\ & 5x_1 + 3x_2 \leq 20 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \text{ integer} \end{aligned}$ | 18.e) | $\begin{aligned} & \max -\lambda x_1 - 5\lambda x_2 + 20\lambda \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 5x_1 + 3x_2 \leq 20 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \text{ integer} \end{aligned}$ |
| 18.c) | $\begin{aligned} & \max(1 - \lambda)x_1 + (4 - 5\lambda)x_2 - 20\lambda \\ \text{s.t.} \quad & x_1, x_2 \geq 0 \text{ integer} \end{aligned}$ | 18.f) | $\begin{aligned} & \max 5x_1 + 7\lambda x_2 + 20\lambda \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 5x_1 + 3x_2 \leq 20 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \text{ integer} \end{aligned}$ |

Lagrangian dual problem (text question)

Q 19: Considering the lagrangian relaxation from the previous question, draw the lagrangian dual problem as function of λ in the interval $0 \leq \lambda \leq 1$. Indicate for which values of λ the slope changes.

Strip packing problem (text question)



The strip packing problem asks for an arrangement of n rectangles into a sheet of minimum width. Rectangle i has width w_i and height h_i . The sheet has width W and height H , where W is to be minimised and H is a constant. There is a number of constraints to be observed: no two rectangles may overlap, and no rectangle may be placed outside the sheet.

We introduce the variables (x_i, y_i) to denote the position of the lower left corner of rectangle i . Moreover, for $i < j$, we introduce some relations $\ell_{ij}, r_{ij}, a_{ij}, b_{ij}$ to denote whether rectangle i is *left*, *right*, *above* or *below* rectangle j . To avoid overlaps, at least one of these relations must hold true for each pair of rectangles. Depending on the relative position of the rectangles, we must demand that

$$\begin{aligned}
 \ell_{ij} = 1 &\Rightarrow x_i + w_i \leq x_j \\
 r_{ij} = 1 &\Rightarrow x_j + w_j \leq x_i \\
 b_{ij} = 1 &\Rightarrow y_i + h_i \leq y_j \\
 a_{ij} = 1 &\Rightarrow y_j + h_j \leq y_i
 \end{aligned} \tag{1}$$

Q 20: Formulate the problem as a mixed-integer linear programming model. You may assume that $W \leq V$, where V is a given upper bound (constant).

Answers

Answer 11 Constraint 11.d) saying $x_1 + x_4 + x_5 \leq 2$ is not a minimal cover inequality since $x_4 + x_5$ is also a cover. All the other constraints are minimal cover inequalities. ■

Answer 12 The constraint

$$x_2 + x_3 + x_4 + \alpha x_5 \leq 2 \quad (2)$$

is valid if it is satisfied for all binary values of x_1, x_2, x_3, x_4, x_5 satisfying

$$3x_1 + 7x_2 + 8x_3 + 6x_4 + 10x_5 \leq 15 \quad (3)$$

If we set $x_5 = 1$ and $x_1 = 0$ then we demand that $x_2 + x_3 + x_4 + \alpha x_5 \leq 2$ is valid when $7x_2 + 8x_3 + 6x_4 + 10 \leq 15$. The largest value of $x_2 + x_3 + x_4$ subject to the given constraint is found by solving the following maximization problem

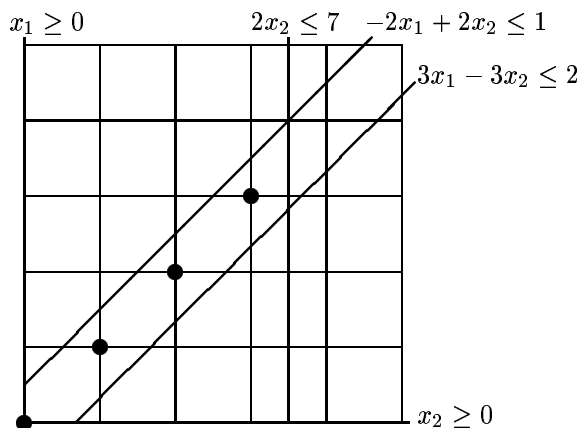
$$\begin{aligned} \text{maximize } \gamma &= x_2 + x_3 + x_4 \\ \text{subject to } &7x_2 + 8x_3 + 6x_4 \leq 5 \\ &x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

which has the solution $\gamma = 0$. Hence knowing that $x_2 + x_3 + x_4$ never will exceed 0 we may set $\alpha = 2 - \gamma = 2$.

Now, setting $x_5 = 0$ we notice that the resulting constraint (2) is trivially satisfied, as it was given in advance.

Setting $x_1 = 1$ we notice that (3) gets tighter, and hence (2) is valid whenever (3) is satisfied. Hence answer 12.c) is correct. ■

Answer 13 As $x_3 \geq 0$ and $x_3 \leq \frac{1}{2}$ and x_3 is integer, then we may conclude that $x_3 = 0$. The remaining problem has two variables and hence may be depicted as follows



The convex hull enclosing the integer solutions (the black points) is a line, and hence it is only possible to find two affine independent points in P . The dimension is $\dim(P) = 1$ and thus the answer is 13.b). ■

Answer 14 We notice that the constraint matrix is totally unimodular, as every row contains at most two nonzero coefficients, and all coefficients are $0, 1, -1$. Placing columns 1 and 3 into set P_1 and columns 2 and 4 into set P_2 we see that property P is satisfied.

Now, knowing that the constraint matrix is totally unimodular, and noting that the right-hand sides are all integers, an optimal solution must be integral (if it exists). As the coefficients in the

objective function are all integers, the solution value will also be integral. The only proposed integer solution is 14.e).

One could also solve the problem using the simplex algorithm, getting the solution $x_1 = 7$, $x_2 = 2$, $x_3 = x_4 = 0$ with objective value $z = 60$. ■

Answer 15 The first inequality with a fractional solution value is

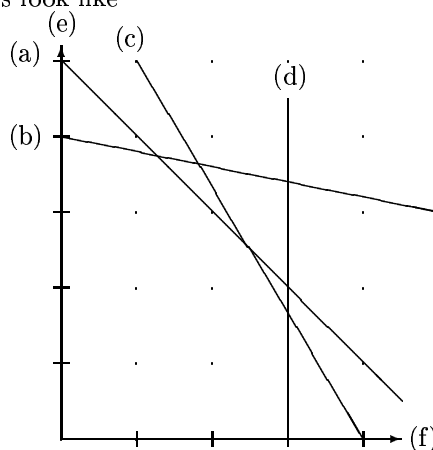
$$\frac{8}{5}x_1 + s_1 + \frac{6}{5}s_2 - \frac{28}{5}s_3 = \frac{136}{5}$$

The Gomory cut becomes

$$\frac{3}{5}x_1 + 0s_1 + \frac{1}{5}s_2 + \frac{2}{5}s_3 \geq \frac{1}{5}$$

hence 15.c) is correct. ■

Answer 16 The constraints look like



It is easily seen that inequalities (d), (e), (f) are facet defining and inequality (a) is redundant. Hence the correct answer is 16.b). ■

Answer 17 Using the proposed multipliers from answer 17.f) we get

$$\begin{array}{rcl} \frac{3}{4}x_1 + \frac{3}{4}x_2 & \leq & \frac{15}{4} \\ \frac{1}{4}x_1 + \frac{5}{4}x_2 & \leq & \frac{20}{4} \\ \hline x_1 + 2x_2 & \leq & \frac{35}{4} \end{array}$$

rounding down the coefficients on both sides we get

$$x_1 + 2x_2 \leq 8$$

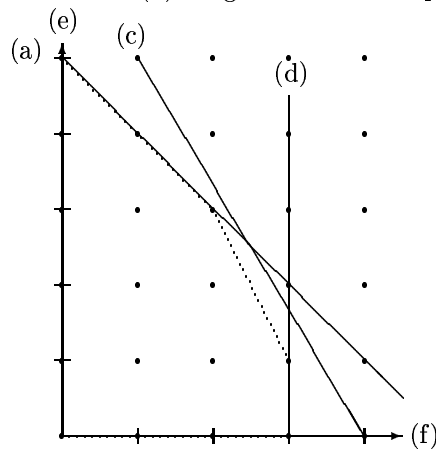
■

Answer 18 Lagrangian relaxing constraint (b) using multiplier $\lambda \geq 0$ we get

$$\begin{array}{ll} \text{maximize} & x_1 + 4x_2 - \lambda(x_1 + 5x_2 - 20) \\ \text{subject to} & x_1 + x_2 \leq 5 \\ & 5x_1 + 3x_2 \leq 20 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \text{ integer} \end{array}$$

reducing the objective function, one gets the problem in 18.a). ■

Answer 19 Having relaxed constraint (b) we get the solution space



where the dotted lines mark the convex hull. The lagrangian dual function has discontinuities when the optimal solution (x_1, x_2) of the lagrangian relaxed problem change. This happens when the isoprofit line is parallel to one of the sides of the convex hull, i.e. when $\lambda = \frac{3}{4}$, $\lambda = \frac{7}{9}$, $\lambda = \frac{4}{5}$ and when $\lambda = 1$.

The objective function at these points is

$$\begin{aligned} \lambda = 0 & : x_1 + 4x_2 \\ \lambda = \frac{3}{4} & : \frac{1}{4}x_1 + \frac{1}{4}x_2 + 15 \\ \lambda = \frac{7}{9} & : \frac{2}{9}x_1 + \frac{1}{9}x_2 + \frac{140}{9} \\ \lambda = \frac{4}{5} & : \frac{1}{5}x_1 + 16 \\ \lambda = 1 & : -x_2 + 20 \end{aligned}$$

For $\lambda = 0$, the optimal solution is $(0, 5)$ having objective 20.00.

For $\lambda = \frac{3}{4}$, one of the optimal solutions is $(2, 3)$ having objective $16 + \frac{1}{4} = 16.25$.

For $\lambda = \frac{7}{9}$, one of the optimal solutions is $(3, 1)$ having objective $16 + \frac{1}{3} = 16.33$.

For $\lambda = \frac{4}{5}$, one of the optimal solutions is $(3, 0)$ having objective $16 + \frac{3}{5} = 16.60$.

For $\lambda = 1$, one of the optimal solutions is $(0, 0)$ having objective 20.00.

Having these breakpoints it is easy to draw the lagrangian dual function in the interval 0 to 1. ■

Answer 20 To formulate the problem as a MIP-model we introduce the variables

- $\ell_{ij} = 1$ iff rectangle i is located left to j
- $r_{ij} = 1$ iff rectangle i is located right to j
- $b_{ij} = 1$ iff rectangle i is located below j
- $a_{ij} = 1$ iff rectangle i is located above j
- (x_i, y_i) are the lower left coordinates of rectangle i .

To ensure that no two rectangles overlap, we demand that

$$\ell_{ij} + r_{ij} + a_{ij} + b_{ij} \geq 1 \quad i, j \in I, i < j$$

Depending on the relative position, the coordinates must satisfy

$$\begin{aligned} \ell_{ij} = 1 & \Rightarrow x_i + w_i \leq x_j \\ r_{ij} = 1 & \Rightarrow x_j + w_j \leq x_i \\ b_{ij} = 1 & \Rightarrow y_i + h_i \leq y_j \\ a_{ij} = 1 & \Rightarrow y_j + h_j \leq y_i \end{aligned}$$

An upper bound M on $(x_i + w_i) - x_j$ is $M = W \leq V$. Hence we may model the first constraint as

$$x_i - x_j + V\ell_{ij} \leq V - w_i$$

The following constraints are modelled in a similar way.

No part of the rectangles may exceed the sheet, hence we demand that $0 \leq x_i \leq W - w_i$ and $0 \leq y_i \leq H - h_i$.

The problem can now be formulated as the following MIP problem

$$\begin{aligned}
\min \quad & W \\
\text{s.t.} \quad & \ell_{ij} + r_{ij} + b_{ij} + a_{ij} \geq 1 && i, j \in I, i < j \\
& x_i - x_j + V\ell_{ij} \leq V - w_i && i, j \in I, i < j \\
& x_j - x_i + Vr_{ij} \leq V - w_j && i, j \in I, i < j \\
& y_i - y_j + Hb_{ij} \leq H - h_i && i, j \in I, i < j \\
& y_j - y_i + Ha_{ij} \leq H - h_j && i, j \in I, i < j \\
& x_i \geq 0 && i \in I \\
& W - x_i \geq w_i && i \in I \\
& y_i \geq 0 && i \in I \\
& y_i \leq H - h_i && i \in I \\
& \ell_{ij}, r_{ij}, b_{ij}, a_{ij} \in \{0, 1\} && i, j \in I, i < j
\end{aligned} \tag{4}$$

■