Using boolean variables to model different constraints (“Snydearket”)

logical expression X	IP model	meaning of constraint
$x > 0 \Leftrightarrow \delta = 1$	$x - \varepsilon\delta \geq 0$ $x - M\delta \leq 0$ (M u.b. on x)	$\delta = 1 \Rightarrow x \geq \varepsilon$ $x > 0 \Rightarrow \delta = 1$
$\sum_{j=1}^{n} a_j x_j \leq b \Leftrightarrow \delta = 1$	$\sum_{j=1}^{n} a_j x_j + M\delta \leq M + b$ $\sum_{j=1}^{n} a_j x_j - (m - \varepsilon)\delta \geq b + \varepsilon$ (M u.b. on $\sum_{j=1}^{n} a_j x_j - b$) ($m$ l.b. on $\sum_{j=1}^{n} a_j x_j - b$)	$\delta = 1 \Rightarrow \sum_{j=1}^{n} a_j x_j \leq b$ $\sum_{j=1}^{n} a_j x_j \leq b \Rightarrow \delta = 1$
$\sum_{j=1}^{n} a_j x_j \geq b \Leftrightarrow \delta = 1$	$\sum_{j=1}^{n} a_j x_j + m\delta \geq m + b$ $\sum_{j=1}^{n} a_j x_j - (M + \varepsilon)\delta \leq b - \varepsilon$ (M u.b. on $\sum_{j=1}^{n} a_j x_j - b$) ($m$ l.b. on $\sum_{j=1}^{n} a_j x_j - b$)	$\delta = 1 \Rightarrow \sum_{j=1}^{n} a_j x_j \geq b$ $\sum_{j=1}^{n} a_j x_j \geq b \Rightarrow \delta = 1$
$(\delta_1 = 1 \land \delta_2 = 1) \Leftrightarrow \delta = 1$	$\delta_1 + \delta_2 - 2\delta \geq 0$ $\delta_1 + \delta_2 - \delta \leq 1$	$\delta = 1 \Rightarrow \delta_1 = 1 \land \delta_2 = 1$ $\delta_1 = 1 \land \delta_2 = 1 \Rightarrow \delta = 1$
$(\delta_1 = 1 \lor \delta_2 = 1) \Leftrightarrow \delta = 1$	$\delta_1 + \delta_2 - \delta \geq 0$ $\delta_1 + \delta_2 - 2\delta \leq 0$	$\delta = 1 \Rightarrow \delta_1 = 1 \lor \delta_2 = 1$ $\delta_1 = 1 \lor \delta_2 = 1 \Rightarrow \delta = 1$
$(\delta_1 = 1 \Rightarrow \delta_2 = 1) \Leftrightarrow \delta = 1$	$\delta_1 - \delta_2 + \delta \leq 1$ $\delta_1 - \delta_2 + 2\delta \geq 1$	$\delta = 1 \Rightarrow (\delta_1 = 1 \Rightarrow \delta_2 = 1)$ $(\delta_1 = 1 \Rightarrow \delta_2 = 1) \Rightarrow \delta = 1$
$\neg(\delta_1 = 1) \Leftrightarrow \delta = 1$	$\delta = 1 - \delta_1$	

Logical conditions may be modeled by associating an indicator variable δ_i with every condition X_i such that

$$\delta_i = 1 \Leftrightarrow X_i = \texttt{true}$$

In this way we may formulate

$X_1 \lor X_2$	$\delta_1 + \delta_2 \geq 1$
$X_1 \land X_2$	$\delta_1 = 1, \delta_2 = 1$
$X_1 \Rightarrow X_2$	$\delta_1 - \delta_2 \leq 0$
$X_1 \Leftrightarrow X_2$	$\delta_1 - \delta_2 = 0$

The equations in the right side of the table can either be added directly to the model, or they can be used to trigger a new indicator variable δ which can be used in other parts of the model.

Notice that in LP and MIP constraints implicitly are linked by an “and”, i.e. all the constraints must be satisfied. This however means that conditions linked by an “and” are much easier to model than those linked by an “or”. In many situations it may be fruitful to rewrite an expression to an equivalent “and” form as illustrated in the following example:

$$(X_1 \lor X_2) \Rightarrow (X_3 \land X_4)$$

can be rewritten to

$$(X_1 \Rightarrow X_3) \land (X_1 \Rightarrow X_4) \land (X_2 \Rightarrow X_3) \land (X_2 \Rightarrow X_4)$$

Introducing an indicator variable δ_i with each of the logical conditions X_i we get the constraints

$$\delta_1 - \delta_3 \leq 0, \quad \delta_1 - \delta_4 \leq 0, \quad \delta_2 - \delta_3 \leq 0, \quad \delta_2 - \delta_4 \leq 0$$

Notice: Assume that we wish to model $\boxed{\delta_i = 1 \Leftrightarrow X_i = \texttt{true}}$.

If δ_i is maximized in the objective function then it is often sufficient to ensure $\boxed{\delta_i = 1 \Rightarrow X_i = \texttt{true}}$.

If δ_i is minimized in the objective function then it is often sufficient to ensure $\boxed{\delta_i = 0 \Rightarrow X_i = \texttt{false}}$.