Wednesday, October 27

Program of the day

- Overview of course, exercises, P2
- Introduction to Integer Programming
- Modelling (Williams, chapter 9)
- Applications: Opencast mining
- Ladies or tigers

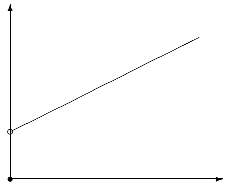
Purpose of the course

- To learn to build complex models from real life using Mathematical Programming
- To know techniques for solving Mathematical Programming models
- To understand that some problems can be solved efficiently and some cannot
- To learn that the same problem may be formulated in different ways, which are easier/harder to solve
- To know a number of techniques for decreasing solution times (or turn a problem from practically “unsolvable” to “solvable”)

Integer Programming

In first part of course: continuous variables, linear constraints

- Most products are integral (apart from liquids)
  Airplane production, Tomato Soups
- Structure of problem leads to IP
  Graph problems
- Nonlinear objective functions or constraints occur frequently



- Logical conditions
  “If I use vegetable oil in the blend, then I must also add 5ml of preservatives”

Integer Programming

General formulation:

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{1j} x_j \leq b_1 \\ & \vdots \\ & \sum_{j=1}^{n} a_{mj} x_j \leq b_m \\ & x_j \geq 0, \quad j = 1, \ldots, n, \qquad x \text{ integer} \end{aligned}$$

where

- A is a $m \times n$ matrix
- b is a m-vector
- c is a n-vector

- IP: integer programming model
- ILP: integer *linear* model (all constraints linear)
- PIP: pure integer programming model
- MIP: mixed integer programming model

Integer Programming, why?

- IP is much more expressible than LP
- As IP is NP-hard, all NP-hard problems can be formulated as IP-models

But

- IP is not an ideal model
- Many problems cannot be formulated as IP-models in a simple way
- All NP-complete problems are “equivalent” and hence equally good
- IP proposed by Edmonds: Expressibility and LP-bounds

Integer Programming

IP powerful method for modelling

- LP easy to solve by e.g. Simplex
  (polynomial time by interior-point methods).
- General IP is NP-hard
- Many concrete problems may be solved despite NP-hardness
- Specific techniques for individual problems

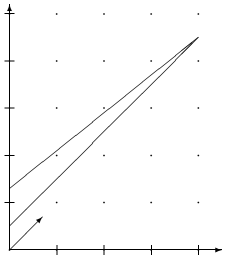
Special problems

- travelling salesman problem
- project selection
- transportation problem
- assignment problem
- assembly line balancing
- set partitioning problem
- aircrew scheduling
- depot location problem
- sequencing problem
- job-shop scheduling

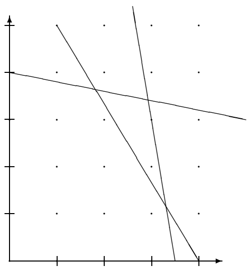
Hardness of IP

$$\begin{array}{rl} \text{maximize} & x_1 + x_2 \\ \text{subject to} & -2x_1 + \quad 2x_2 \geq 1 \\ & -8x_1 + 10x_2 \leq 13 \\ & x_1, x_2 \geq 0, \text{integer} \end{array}$$

Solutions are not found in extreme points (or nearby)



Find convex hull



Model building

- Indicator variables
- Non-convex problems
- Nonlinear fuctions
- Logical expressions
- Transformation of “human text” to ILP

Indicator variables

- Most important modelling tool!
- $\delta \in \{0,1\}$
- $\delta = 1$ if and only if some event happens.

Model:

$$\delta = 1 \ \Leftrightarrow \ x > 0$$
$$\delta \in \{0,1\}, \ \ x \geq 0$$

$\boxed{\delta = 1 \ \Rightarrow \ x > 0}$

$\delta = 1 \Rightarrow x \geq \varepsilon$ ε level for x regarded as 0

$x - \varepsilon\delta \geq 0, \ \delta \in \{0,1\}$

$\boxed{x > 0 \ \Rightarrow \ \delta = 1}$

$\delta = 0 \ \Rightarrow \ x = 0$

$x - M\delta \leq 0, \ \delta \in \{0,1\}$ M upper bound on x

Indicator variables

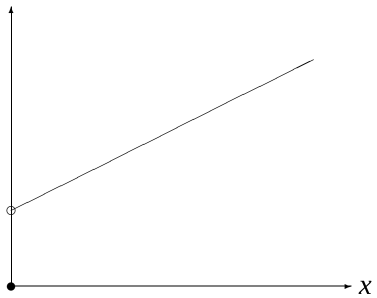
Logical implications $\boxed{X \ \Leftrightarrow \ Y}$

X	Y	$X \Rightarrow Y$	$X \Leftarrow Y$	$\neg X \Rightarrow \neg Y$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Fixed-charge problem

cost function

$$f(x) = \begin{cases} ax + b & \text{if } \ x > 0 \\ 0 & \text{if } \ x = 0 \end{cases}$$



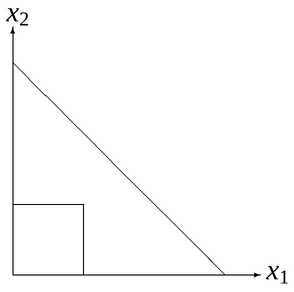
Model:

$$\begin{cases} \text{minimize} & ax + \delta b \\ \text{subject to} & x - M\delta \leq 0 \\ & x - \varepsilon\delta \geq 0 \\ & \delta \in \{0,1\}, \ \ x \geq 0 \end{cases}$$

Non-convex problems

constraints:

$$x_1 + x_2 \leq b$$
$$x_1 \geq 1 \text{ or } x_2 \geq 1$$
$$x_1, x_2 \geq 0$$



Modeling tool

$$\delta = 1 \Rightarrow x \geq \varepsilon$$

Two indicator variables δ_1, δ_2:

$$\begin{cases} x_1 + x_2 & \leq & b \\ \delta_1 + \delta_2 & \geq & 1 \\ x_1 - 1\delta_1 & \geq & 0 \\ x_2 - 1\delta_2 & \geq & 0 \\ x_1, x_2 & \geq & 0, \\ \delta_1, \delta_2 & \in & \{0,1\} \end{cases}$$

Indicator variables

"if A is included in the blend then B is included in the blend"

can be modeled by using constraints

$x_A > 0 \;\Rightarrow\; \delta = 1$	$x_A - M\delta \le 0,\;\; \delta \in \{0,1\}$ M upper bound on x_A
$\delta = 1 \;\Rightarrow\; x_B > 0$	$x_B - \varepsilon\delta \ge 0,\;\; \delta \in \{0,1\}$ ε level for x_B regarded as 0

Example

Assume that x_A and x_B are proportions in blend i.e. $x_A + x_B = 1$.

$$M = 1 \qquad \varepsilon = 0.01$$

Formulation:

$$\begin{cases} x_A - \delta & \le \; 0 \\ x_B - 0.01\delta & \ge \; 0 \\ \delta & \in \; \{0,1\} \end{cases}$$

Indicator variables for linear inequalities

Example

"If resources needed for production of x_1, x_2 and x_3 are below the limit of one truck, then use the other truck for some other purpose."

General form

$$\boxed{\sum_{j=1}^{n} a_j x_j \le b \;\;\Leftrightarrow\;\; \delta = 1, \quad \delta \in \{0,1\}}$$

- $\sum_{j=1}^{n} a_j x_j \le b \Leftarrow \delta = 1$ has the MIP formulation

$$\boxed{\sum_{j=1}^{n} a_j x_j + M\delta \le M + b}$$

where M is upper bound on $\sum_{j=1}^{n} a_j x_j - b$

$\delta = 1$: $\sum_{j=1}^{n} a_j x_j \le b$

$\delta = 0$: $\sum_{j=1}^{n} a_j x_j - b \le M$

Indicator variables for linear inequalities

- $\sum_{j=1}^{n} a_j x_j \le b \Rightarrow \delta = 1$ has the MIP formulation

$$\boxed{\sum_{j=1}^{n} a_j x_j - (m - \varepsilon)\delta \ge b + \varepsilon}$$

where m is lower bound on $\sum_{j=1}^{n} a_j x_j - b$.

$$\delta = 0 \Rightarrow \sum_{j=1}^{n} a_j x_j \ge b + \varepsilon$$

$\delta = 0$: $\sum_{j=1}^{n} a_j x_j \ge b + \varepsilon$

$\delta = 1$: $\sum_{j=1}^{n} a_j x_j - m + \varepsilon \ge b + \varepsilon$

$\sum_{j=1}^{n} a_j x_j - b \ge m$

Indicator variables for inequalities, example

Logical condition

$$\begin{aligned} &2x_1 + 3x_2 \le 1 \Leftrightarrow \delta = 1 \\ &\delta \in \{0,1\} \\ &0 \le x_1 \le 1, \;\; 0 \le x_2 \le 1 \end{aligned}$$

We find

$$\begin{aligned} M &= \text{u.b.}(\textstyle\sum_{j=1}^{n} a_j x_j - b) \\ &= \text{u.b.}(2x_1 + 3x_2 - 1) = 4 \end{aligned}$$

and

$$\begin{aligned} m &= \text{l.b.}(\textstyle\sum_{j=1}^{n} a_j x_j - b) \\ &= \text{l.b.}(2x_1 + 3x_2 - 1) = -1 \end{aligned}$$

choose $\varepsilon = 0.01$, i.e. constraint broken when $2x_1 + 3x_2 \ge 1.01$

Constraints

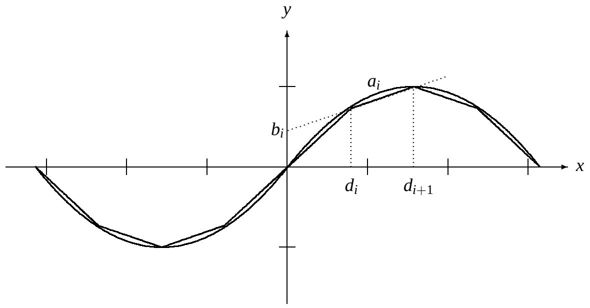
$$\begin{aligned} &2x_1 + 3x_2 + 4\delta \le 4 + 1 \\ &2x_1 + 3x_2 - (-1 - 0.01)\delta \ge 1 + 0.01 \end{aligned}$$

Which results in model:

$$\begin{cases} 2x_1 + 3x_2 + 4\delta \le 5 \\ 2x_1 + 3x_2 + 1.01\delta \ge 1.01 \\ \delta \in \{0,1\} \\ 0 \le x_1 \le 1, \\ 0 \le x_2 \le 1 \end{cases}$$

Nonlinear functions

Frequently, the objective function or some of the constraints may contain nonlinear functions.



Approx. nonlinear function by piecewise linear function

- Split into m intervals
- For each interval $[d_i, d_{i+1}]$

$$d_i \leq x \leq d_{i+1} \;\Leftrightarrow\; y = a_i x + b_i$$

- Model as

$$\begin{cases} d_i \leq x & \Leftrightarrow \quad \delta_1 = 1 \\ x \leq d_{i+1} & \Leftrightarrow \quad \delta_2 = 1 \\ \delta_1 + \delta_2 = 2 & \Leftrightarrow \quad \delta = 1 \\ y = a_i x + b_i & \Leftrightarrow \quad \delta = 1 \end{cases}$$

- Many intervals m, better precision
  but much harder to solve!

Logical conditions and 0-1 variables

a) *If no depot is sited here then it will not be possible to supply any of the customers from the depot.*

b) *If we manufacture product A then we must also manufacture product B or at least one of products C and D.*

c) *If we do not place an electronic module in this position, then no wires can be connected into this position.*

Introduce an indicator variable $\delta_i \in \{0,1\}$ with each condition X_i

$$\boxed{\text{condition } X_i \text{ is true} \Leftrightarrow \delta_i = 1}$$

In this way we may formulate:

$X_1 \vee X_2$	$\delta_1 + \delta_2 \geq 1$
$X_1 \wedge X_2$	$\delta_1 = 1, \delta_2 = 1$
$X_1 \Rightarrow X_2$	$\delta_1 - \delta_2 \leq 0$
$X_1 \Leftrightarrow X_2$	$\delta_1 - \delta_2 = 0$

Transformation to linear form

Write up the text in ordinary mathematical form

$$(\sin(x_1) \leq \tfrac{1}{2} \;\vee\; x_1 x_2 \leq x_3) \Rightarrow (x_3 = 1 \vee x_2 + x_1 \leq 1)$$

Stepwise transformation

1 Arithmetic functions are replaced by piecewise linear approximations of the functions.

2 Products of decision variables are transformed into products of binary variables. Products of binary variables may easily be expressed as logical constraints, and thus put on binary form.

3 Relations are transformed into linear inequalities with boolean variables.

$$(ax \leq b) \Leftrightarrow (\delta = 1)$$

4 Boolean logics are transformed into linear form.

$$(B_1 \vee B_2) \Leftrightarrow (\delta' = 1)$$

5 The resulting expression should be true $\delta_{all} = 1$

6 Domains of variables are defined.

$$\delta_i \in \{0,1\}$$

Transformation of general constraints to linear form

Step 3:

Relation	ILP-constraints
$Ax \leq b$	$Ax + (\delta - 1)M \leq b, \quad Ax + \delta M \geq b + \varepsilon$
$Ax < b$	$Ax + (\delta - 1)M \leq b - \varepsilon, \quad Ax + \delta M \geq b$
$Ax > b$	$Ax + (1 - \delta)M \geq b + \varepsilon, \quad Ax - \delta M \leq b$
$Ax \geq b$	$Ax + (1 - \delta)M \geq b, \quad Ax - \delta M \leq b - \varepsilon$
$Ax = b$	$Ax \geq b \;\wedge\; Ax \leq b,$

Step 4:

Relation	Meaning	ILP-constraints
$B_1 \vee B_2$	$\delta = 1 \Leftrightarrow \delta_1 = 1 \vee \delta_2 = 1$	$\delta - \delta_1 - \delta_2 \leq 0, \;\; \delta_1 + \delta_2 - 2\delta \leq 0$
$B_1 \wedge B_2$	$\delta = 1 \Leftrightarrow \delta_1 = 1 \wedge \delta_2 = 1$	$2\delta - \delta_1 - \delta_2 \leq 0, \;\; \delta_1 + \delta_2 - \delta \leq 1$
$B_1 \Rightarrow B_2$	$\delta = 1 \Leftrightarrow (\delta_1 = 1 \Rightarrow \delta_2 = 1)$	$\delta_1 - \delta_2 + \delta \leq 1, \;\; \delta_1 - \delta_2 + 2\delta \geq 1$
$B_1 \Leftrightarrow B_2$	$\delta = 1 \Leftrightarrow (\delta_1 = 1 \Leftrightarrow \delta_2 = 1)$	use: $(B_1 \Rightarrow B_2) \wedge (B_2 \Rightarrow B_1)$
$\neg B_1$	$\delta = 1 \Leftrightarrow \neg(\delta_1 = 1)$	$\delta = 1 - \delta_1$

Opencast mining (williams example 12.15)

Concentration c_{ijk} of pure metal for each block

- Level 1 (surface)

1.5	1.5	1.5	0.75
1.5	2.0	1.5	0.75
1.0	1.0	0.75	0.50
1.5	1.5	1.5	0.25

- Level 2 (25 ft depth)

4.0	4.0	2.0
3.0	3.0	1.0
2.0	2.0	0.5

- Level 3 (50 ft depth)

12.0	6.0
5.0	4.0

- Level 4 (75 ft depth)

6.0

Cost of extraction

Level 1: $e_1 = 3.000$ pounds
Level 2: $e_2 = 6.000$ pounds
Level 3: $e_3 = 8.000$ pounds
Level 4: $e_4 = 10.000$ pounds

Revenue from 100% block is $R = 200.000$ pounds

Opencast mining

Introduce variables δ_{ijk}

- Level 1 (surface)

δ_{111}	δ_{112}	δ_{113}	δ_{114}
δ_{121}	δ_{122}	δ_{123}	δ_{124}
δ_{131}	δ_{132}	δ_{133}	δ_{134}
δ_{141}	δ_{142}	δ_{143}	δ_{144}

- Level 2 (25 ft depth)

δ_{211}	δ_{212}	δ_{213}
δ_{221}	δ_{222}	δ_{223}
δ_{231}	δ_{232}	δ_{233}

- Level 3 (50 ft depth)

δ_{311}	δ_{312}
δ_{321}	δ_{322}

- Level 4 (75 ft depth)

δ_{411}

Maximize net profit

$$\sum (R \cdot c_{ijk} - e_i)\delta_{ijk}$$

Constraints

$$\delta_{ijk} = 1 \Rightarrow \begin{cases} \delta_{i-1,j,k} & = 1 \\ \delta_{i-1,j+1,k} & = 1 \\ \delta_{i-1,j,k+1} & = 1 \\ \delta_{i-1,j+1,k+1} & = 1 \end{cases}$$

All variables $\delta_{ijk} \in \{0, 1\}$.

Opencast mining

The constraints

$$\delta_{ijk} = 1 \Rightarrow \begin{cases} \delta_{i-1,j,k} & = 1 \\ \delta_{i-1,j+1,k} & = 1 \\ \delta_{i-1,j,k+1} & = 1 \\ \delta_{i-1,j+1,k+1} & = 1 \end{cases}$$

can be expressed as

$$\begin{aligned} \delta_{ijk} - \delta_{i-1,j,k} & \leq 0 \\ \delta_{ijk} - \delta_{i-1,j+1,k} & \leq 0 \\ \delta_{ijk} - \delta_{i-1,j,k+1} & \leq 0 \\ \delta_{ijk} - \delta_{i-1,j+1,k+1} & \leq 0 \end{aligned}$$

Model with 30 binary variables, 56 constraints
Brute-force solution: 2^{30} steps

Opencast mining

$$\begin{pmatrix} -1 & & & & & 1 & & & & & & \\ & -1 & & & & 1 & & & & & & \\ & & -1 & & & 1 & & & & & & \\ & & & -1 & & 1 & & & & & & \\ & -1 & & & & & 1 & & & & & \\ & & -1 & & & & 1 & & & & & \\ & & & -1 & & & 1 & & & & & \\ & & & & -1 & & 1 & & & & & \\ & & & & & & & & & & & \\ & & & & & & & -1 & & & & 1 \\ & & & & & & & & -1 & & & 1 \\ & & & & & & & & & -1 & & 1 \\ & & & & & & & & & & -1 & 1 \end{pmatrix}$$

The constraint matrix A is totally unimodular (TU)

Solving

$$\begin{array}{ll} \max & c\delta \\ \text{s.t.} & A\delta \leq b \\ & \delta \geq 0 \end{array}$$

gives integer solutions for any integer vector b and any c

Opencast mining

The optimal solution

- Level 1 (surface)

■	■	■	
■	■	■	
■	■	■	

- Level 2 (25 ft depth)

■	■	
■	■	

- Level 3 (50 ft depth)

■	

- Level 4 (75 ft depth)

Net profit is 17.500 pounds.

Ladies or tigers (Smullyan 1982)

Prisoner in castle, meets nine doors:

- Lady (immediately marry her)
- Tiger (immediately eaten)

Prefers to marry lady than be eaten

Statements on the nine doors are:

Door 1: The lady is in an odd-numbered room
Door 2: This room is empty
Door 3: Either sign 5 is right or sign 7 is wrong
Door 4: Sign 1 is wrong
Door 5: Either sign 2 or sign 4 is right
Door 6: Sign 3 is wrong
Door 7: The lady is not in room 1
Door 8: This room contains a tiger and room 9 is empty
Door 9: This room contains a tiger and sign 6 is wrong

Ladies or tigers (Smullyan 1982)

In addition

- Only one lady
- Each other room: Tiger or empty
- Sign on door of lady is true
- Sign on door of every tiger is false
- Sign on door of empty room can be either true or false.

No unique solution until the prisoner is told whether or not room eight is empty. Then unique solution

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Ladies or tigers (Smullyan 1982)

Define subscripts $i = 1, \ldots, 9$ and $j = 1, \ldots, 3$ where (1 lady, 2 tiger, 3 empty) and as above variables are

$$x_{i,j} = \begin{cases} 1 & \text{if door } i \text{ hides prize } j \\ 0 & \text{otherwise} \end{cases}$$

$$t_i = \begin{cases} 1 & \text{if statement on door } i \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

“Either-or” means “ordinary or”