Example: Prize Collecting Traveling Salesman Problem

- Set of $N$ cities.
- Salesman starts in city 1.
- To each edge $e$ is associated a cost $c_e$.
- To each node $j$ is associated a profit $f_j$.
- Visit at least two other cities.
- Maximize profit — cost.

Introduce variables

- $x_e = 1$ if edge $e$ is used.
- $y_j = 1$ if node $j$ is visited.

Formulation

$$\text{max } \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e$$

s.t. $$\sum_{e \in E(j)} x_e = 2y_j, \ j \in N$$

$$y_1 = 1$$

$$x \in \{0,1\}, y \in \{0,1\}$$

Separation for generalized subtour constraints

Assume that we solve the ILP-problem

$$\text{max } \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e$$

s.t. $$\sum_{e \in E(j)} x_e = 2y_j, \ j \in N$$

$$y_1 = 1$$

$$x \in \{0,1\}, y \in \{0,1\}$$

getting a solution $(x^*, y^*)$. How do we find a violated GSE constraint?

- $N' = N \setminus 1$
- $E' = E \setminus \{\delta(1)\}$
- $z_i = 1$ iff $i \in S$

A constraint for $(k, S)$ is violated if

$$\sum_{e \in E'(S)} x_e^* z_i^* > \sum_{i \in S \setminus \{k\}} y_i^* z_i^*$$

For each $k$ this can be formulated as IP-model

$$\gamma = \text{max } \sum_{e = (i, j) \in E'} x_e^* z_i^* - \sum_{i \in N \setminus \{k\}} y_i^* z_i$$

s.t. $$z_k = 1$$

$$z \in \{0,1\}$$

Separation for generalized subtour constraints

The quadratic 0-1 program

$$\gamma = \text{max } \sum_{e = (i, j) \in E'} x_e^* z_i^* - \sum_{i \in N \setminus \{k\}} y_i^* z_i$$

s.t. $$z_k = 1$$

$$z \in \{0,1\}$$

can be reformulated using

$$w_{(i, j)} = 1 \iff z_i = 1 \text{ and } z_j = 1$$

but since we maximize only

$$w_{(i, j)} = 1 \Rightarrow z_i = 1 \text{ and } z_j = 1$$

is needed

$$\gamma = \text{max } \sum_{e = (i, j) \in E'} x_e^* w_e - \sum_{i \in N \setminus \{k\}} y_i^* z_i$$

s.t. $$w_{(i, j)} \leq z_i, (i, j) \in E'$$

$$w_{(i, j)} \leq z_j, (i, j) \in E'$$

$$z_k = 1$$

$$z \in \{0,1\}$$

This formulation is TU and thus can be solved in polynomial time.
Separation for generalized subtour constraints

\[ f = (2, 4, 1, 3, 7, 1, 7) \text{ and } c_e = \begin{pmatrix}
2 & 3 & 5 & 2 & 5 \\
-5 & 3 & 3 & 4 & 7 \\
-4 & 6 & 0 & 4 \\
-4 & -4 & -5 & 8 \\
-3 \\
-3 \\
\end{pmatrix} \]

The LP-relaxation of (1) gives the routes

\((1, 5, 2, 4)\) and \((3, 6, 7)\)

The separation algorithm returns

\[ x_{36} + x_{37} + x_{67} \leq y_3 + y_7 \]

which cuts off the subtour \((3, 6, 7)\).

Relaxation

In a branch-and-bound algorithm we find upper bounds by relaxing the problem

Relaxation (Wolsey sec. 2.1)

\[
\begin{align*}
\text{max} \{ cx : x \in S \} & \quad (IP) \\
\text{max} \{ f(x) : x \in T \} & \quad (RP)
\end{align*}
\]

RP is a relaxation of IP if

- \( S \subseteq T \)
- \( f(x) \geq cx \) for all \( x \in S \)

Which constraints should be relaxed?

- Quality of bound (tightness of relaxation)
- Remaining problem can be solved efficiently
- Proper multipliers can be found efficiently
- Constraints difficult to formulate mathematically
- Constraints which are too expensive to write up

Overview

Different relaxations
- LP-relaxation
- Deleting constraint
- Lagrangian relaxation
- Surrogate relaxation
- Semidefinite relaxation

Relaxations are often used in combination.

Hierarchy

- Best surrogate relaxation
- Best lagrangian relaxation
- LP-relaxation

Lagrangian relaxation, example

\[
\begin{align*}
\text{maximize} & \quad 4x_1 + x_2 \\
\text{subject to} & \quad 3x_1 - x_2 \leq 6 \\
& \quad x_2 \leq 3 \\
& \quad 5x_1 + 2x_2 \leq 18 \\
& \quad x_1, x_2 \geq 0, \text{ integer}
\end{align*}
\]

IP solution \((x_1, x_2) = (2, 3)\) with \(z_{IP} = 11\)

LP solution \((x_1, x_2) = \left(\frac{30}{11}, \frac{24}{11}\right)\) with \(z_{LP} = \frac{144}{11} = 13.1\)

Last constraint complicating, relax using multiplier \(\lambda = \frac{1}{2}\)

\[
\begin{align*}
\text{maximize} & \quad 4x_1 + x_2 - \frac{1}{2}(5x_1 + 2x_2 - 18) = \frac{3}{2}x_1 + 9 \\
\text{subject to} & \quad 3x_1 - x_2 \leq 6 \\
& \quad x_2 \leq 3 \\
& \quad x_1, x_2 \geq 0, \text{ integer}
\end{align*}
\]

Solution \((x_1, x_2) = (3, 3)\) with \(z_{LR} = \frac{3}{2}3 + 9 = 13.5\)

Upper bound
Lagrangian relaxation

Integer Programming Problem

\[
\begin{align*}
\text{maximize} & \quad cx \\
\text{subject to} & \quad Ax \leq b \\
& \quad Dx \leq d \\
& \quad x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

Lagrange relax \( Dx \leq d \), using multipliers \( \lambda \geq 0 \)

\[
\begin{align*}
\text{maximize} & \quad z_{LR}(\lambda) = cx - \lambda(Dx - d) \\
\text{subject to} & \quad Ax \leq b \\
& \quad x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

**Proposition 1** Optimal solution to relaxed problem gives upper bound on original problem

**Proof** show that relaxation

multiplier \( \lambda \), “punishment”

If \( \lambda \) large \( \Rightarrow \) constraint satisfied

If \( \lambda = 0 \) \( \Rightarrow \) drop constrain

---

Geom. interpretation, Lagrangian Relaxation

Original problem, integer solution

\( (x_1, x_2) = (4, 0) \)

\( z = 28.00 \)

Original problem, LP-relaxed solution

\( (x_1, x_2) = \left( \frac{36}{11}, \frac{40}{11} \right) = (3.27, 3.64) \)

\( z = 30.18 \)

Drop first constraint, integer solution

\( (x_1, x_2) = (3, 4) \)

\( z = 29.00 \)

Drop first constraint, LP-relaxed solution

\( (x_1, x_2) = \left( \frac{16}{5}, 4 \right) = (3.2, 4) \)

\( z = 30.40 \)

Maximum on \( Q \), LP-relaxed solution

\( (x_1, x_2) = (3, 4) \)

\( z = 29.00 \)

Maximum on \( Q \), with first constraint added

\( (x_1, x_2) = \left( \frac{28}{9}, \frac{32}{9} \right) = (3.11, 3.56) \)

\( z = 28.88 \)
**Geom. interpretation, Lagrangian Relaxation**

**Viewpoint 1:** fixed $\lambda$

\[
\begin{align*}
\text{max} \quad & (7 + \lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda \\
\text{s.t.} \quad & 5x_1 + x_2 \leq 20 \\
\quad & -2x_1 - 2x_2 \leq -7 \\
\quad & -x_1 \leq -2 \\
\quad & x_2 \leq 4 \\
\quad & x_1, x_2 \text{ integer}
\end{align*}
\]

Redefinition using convex hull of $Q$

\[
\begin{align*}
\text{max} \quad & (7 + \lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda \\
\text{s.t.} \quad & 4x_1 + x_2 \leq 16 \\
\quad & -x_1 - x_2 \leq -4 \\
\quad & -x_1 \leq -2 \\
\quad & x_2 \leq 4 \\
\quad & x_1, x_2 \text{ integer}
\end{align*}
\]

**Geom. interpretation, Lagrangian Relaxation**

\[
\begin{align*}
\lambda & = 0 \\
\lambda & = 1/2 \\
\lambda & = 2 \\
\lambda & = \infty
\end{align*}
\]

- $\lambda$ is a modifier of the objective function.
- For $0 \leq \lambda \leq 1/9$, optimal solution $(3,4)$
  \[z_{LR}(\lambda) = (7 + \lambda)3 + (2 - 2\lambda)4 + 4\lambda = 29 - \lambda\]
- For $\lambda \geq 1$ optimal solution $(4,0)$
  \[z_{LR}(\lambda) = (7 + \lambda)4 + (2 - 2\lambda)0 + 4\lambda = 28 + 8\lambda\]
- Increasing lambda is forcing the optimal solution to satisfy relaxed constraint.

**Geom. interpretation, Lagrangian Relaxation**

**Viewpoint 1:**

\[
\begin{align*}
\text{max} \quad & cx \\
\text{subject to} \quad & Ax \leq b \\
\quad & Dx \leq d \\
\quad & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

\[
\text{max} \left\{ cx : x \in \text{conv}(Ax \leq b, Dx \leq d, x \in \mathbb{Z}_+) \right\}
\]

Lagrange Relaxation, multipliers $\lambda \geq 0$

\[
\begin{align*}
\text{maximize} \quad & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\
\text{subject to} \quad & Ax \leq b \\
\quad & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

for best multiplier $\lambda \geq 0$

\[
\text{max} \left\{ cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+) \right\}
\]

This “proves” theorem 10.3 page 172.
Geom. interpretation, Lagrangian Relaxation

Viewpoint 2: fixed point $x^i$

There are 8 integer points in $Q$:

$$
\{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8\} = \\
\{(2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 0)\}
$$

For fixed $x^i$ the objective function

$$
z_{LR}(\lambda, x^i) = (7 + \lambda)x_1^i + (2 - 2\lambda)x_2^i + 4\lambda = 7x_1^i + 2x_2^i + \lambda(x_1^i - 2x_2^i + 4)
$$

is an affine function.

E.g. for $x^7 = (3, 4)$

$$
z_{LR}(\lambda, x^7) = 7 \cdot 3 + 2 \cdot 4 + \lambda(3 - 2 \cdot 4 + 4) = 29 - \lambda
$$

Lagrangian relaxation and duality

- Lagrangian relaxation is a generalization of duality, where we may “dualize” any subset of constraints.

- Lagrange Relaxation

  maximize $z_{LR}(\lambda) = cx - \lambda(Dx - d)$

  subject to

  $Ax \leq b$

  $x_j \in \mathbb{Z}^+$, $j = 1, \ldots, n$

Lagrangian Dual Problem

$$
z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)
$$

is an LP-problem

- Optimal multipliers $\lambda$ may be found by simplex.

- Subgradient is however faster when few iterations.

Lagrangian Relaxation

Integer Programming Problem

maximize $cx$

subject to

$Ax \leq b$

$Dx \leq d$

$x_j \in \mathbb{Z}^+$, $j = 1, \ldots, n$

Lagrange Relaxation, multipliers $\lambda \geq 0$

maximize $z_{LR}(\lambda) = cx - \lambda(Dx - d)$

subject to

$Ax \leq b$

$x_j \in \mathbb{Z}^+$, $j = 1, \ldots, n$

Lagrangian Dual Problem

$$
z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)
$$

Assume that the “nice constraints” $Ax \leq b$ define the convex hull, e.g.

- $A$ is totally unimodular, and $b$ is a vector of integers

- There are no constraints left

- The remaining constraints are defined in linear variables
Lagrangian Relaxation

for best multiplier $\lambda \geq 0$ strength of model

$$\max \left\{ cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+) \right\}$$

If $\{x : Ax \leq b\} = \{x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+)\}$ strength

$$\max \left\{ cx : Dx \leq d, Ax \leq b \right\}$$

Corollary (page 173 in Wolsey)

$z_{LD} = z_{LP}$

for any objective function $cx$.

• We do not obtain better bounds than by linear relaxation.
• We may find $z_{LP} = z_{LD}$ in polynomial time.
• If the remaining problem $Ax \leq b$ has a nice structure (e.g. min-spanning-tree) we may find $z_{LD}$ faster than $z_{LP}$.

Lagrangian Relaxation

Integer Programming Problem

$$\text{maximize } cx \quad \text{subject to }\begin{array}{l}
Ax \leq b \\
Dx \leq d \\
x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{array}$$

If $Ax \leq b$ define convex hull, solution to Lagrangian dual

$\lambda = y'$

Lagrangian relaxation

• If relax all constraints: ordinary dual problem
• Lagrangian relaxation of a constraint can be seen as “dualization” of a constraint.
• We have found a technique for deriving the best lagrangian multipliers in some special cases.