

Program of the day:

- Example: Prize Collecting Traveling Salesman Problem
- Relaxation strategies.
- Lagrangian relaxation.

- Set of N cities.
- Salesman starts in city 1.
- To each edge e is associated a cost c_e
- To each node j is associated a profit f_j
- Visit *at least* two other cities
- Maximize profit – cost.

Introduce variables

- $x_e = 1$ if edge e is used.
- $y_j = 1$ if node j is visited.

Formulation

$$\begin{aligned} \max \quad & \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(j)} x_e = 2y_j \quad , j \in N \\ & \sum_{e \in E(S)} x_e \leq \sum_{i \in S \setminus \{k\}} y_i \quad , k \in S, S \subseteq N \setminus \{1\} \\ & y_1 = 1 \\ & x_e \in \{0, 1\}, y_j \in \{0, 1\} \end{aligned}$$

Separation for generalized subtour constraints

Assume that we solve the ILP-problem

$$\begin{aligned} \max \quad & \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(j)} x_e = 2y_j \quad , j \in N \\ & y_1 = 1 \\ & x \in \{0, 1\}, y \in \{0, 1\} \end{aligned} \quad (1)$$

getting a solution (x^*, y^*) . How do we find a violated GSE constraint?

- $N' = N \setminus 1$
- $E' = E \setminus \{\delta(1)\}$
- $z_i = 1$ iff $i \in S$

A constraint for (k, S) is violated if

$$\sum_{e \in E'(S)} x_e^* > \sum_{i \in S \setminus \{k\}} y_i^*$$

For each k this can be formulated as IP-model

$$\begin{aligned} \gamma = \max \quad & \sum_{e=(i,j) \in E'} x_e^* z_i z_j - \sum_{i \in N' \setminus \{k\}} y_i^* z_i \\ \text{s.t.} \quad & z_k = 1 \\ & z \in \{0, 1\} \end{aligned}$$

Separation for generalized subtour constraints

The quadratic 0-1 program

$$\begin{aligned} \gamma = \max \quad & \sum_{e=(i,j) \in E'} x_e^* z_i z_j - \sum_{i \in N' \setminus \{k\}} y_i^* z_i \\ \text{s.t.} \quad & z_k = 1 \\ & z \in \{0, 1\} \end{aligned}$$

can be reformulated using

$$w_{(i,j)} = 1 \Leftrightarrow z_i = 1 \text{ and } z_j = 1$$

but since we maximize only

$$w_{(i,j)} = 1 \Rightarrow z_i = 1 \text{ and } z_j = 1$$

is needed

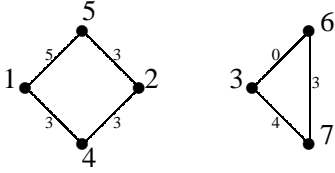
$$\begin{aligned} \gamma = \max \quad & \sum_{e=(i,j) \in E'} x_e^* w_e - \sum_{i \in N' \setminus \{k\}} y_i^* z_i \\ \text{s.t.} \quad & w_{(i,j)} \leq z_i \quad , (i,j) \in E' \\ & w_{(i,j)} \leq z_j \quad , (i,j) \in E' \\ & z_k = 1 \\ & w \in \{0, 1\}, z \in \{0, 1\} \end{aligned}$$

This formulation is TU and thus can be solved in polynomial time

Separation for generalized subtour constraints

$$f = (2, 4, 1, 3, 7, 1, 7) \text{ and}$$

$$c_e = \begin{pmatrix} - & 4 & 3 & 3 & 5 & 2 & 5 \\ - & - & 5 & 3 & 3 & 4 & 7 \\ - & - & - & 4 & 6 & 0 & 4 \\ - & - & - & - & 4 & 4 & 6 \\ - & - & - & - & - & 5 & 8 \\ - & - & - & - & - & - & 3 \\ - & - & - & - & - & - & - \end{pmatrix}$$



The LP-relaxation of (1) gives the routes

$$(1, 5, 2, 4) \text{ and } (3, 6, 7)$$

The separation algorithm returns

$$x_{36} + x_{37} + x_{67} \leq y_3 + y_7$$

which cuts off the subtour (3, 6, 7).

5

Relaxation

In a branch-and-bound algorithm we find upper bounds by relaxing the problem

Relaxation (Wolsey sec. 2.1)

$$\max\{cx : x \in S\} \quad (IP)$$

$$\max\{f(x) : x \in T\} \quad (RP)$$

RP is a relaxation of IP if

- $S \subseteq T$
- $f(x) \geq cx$ for all $x \in S$

Which constraints should be relaxed?

- Quality of bound (tightness of relaxation)
- Remaining problem can be solved efficiently
- Proper multipliers can be found efficiently
- Constraints difficult to formulate mathematically
- Constraints which are too expensive to write up

6

Overview

Different relaxations

- LP-relaxation
- Deleting constraint
- Lagrangian relaxation
- Surrogate relaxation
- Semidefinite relaxation

Relaxations are often used in combination.

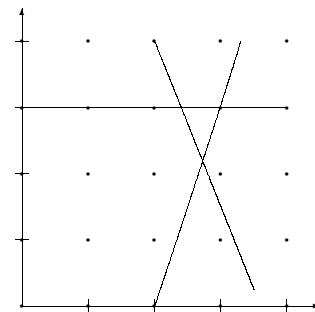
Hierarchy

- Best surrogate relaxation
- Best lagrangian relaxation
- LP-relaxation

7

Lagrangian relaxation, example

$$\begin{aligned} \text{maximize} \quad & 4x_1 + x_2 \\ \text{subject to} \quad & 3x_1 - x_2 \leq 6 \\ & x_2 \leq 3 \\ & 5x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0, \text{ integer} \end{aligned}$$



IP solution $(x_1, x_2) = (2, 3)$ with $z_{IP} = 11$

LP solution $(x_1, x_2) = (\frac{30}{11}, \frac{24}{11})$ with $z_{LP} = \frac{144}{11} = 13.1$

Last constraint complicating, relax using multiplier $\lambda = \frac{1}{2}$

$$\begin{aligned} \text{maximize} \quad & 4x_1 + x_2 - \frac{1}{2}(5x_1 + 2x_2 - 18) = \frac{3}{2}x_1 + 9 \\ \text{subject to} \quad & 3x_1 - x_2 \leq 6 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0, \text{ integer} \end{aligned}$$

Solution $(x_1, x_2) = (3, 3)$ with $z_{LR} = \frac{3}{2} \cdot 3 + 9 = 13.5$
Upper bound

8

Lagrangian relaxation

Integer Programming Problem

$$\begin{aligned} & \text{maximize } cx \\ & \text{subject to } Ax \leq b \\ & \quad Dx \leq d \\ & \quad x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

Lagrange relax $Dx \leq d$, using multipliers $\lambda \geq 0$

$$\begin{aligned} & \text{maximize } z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ & \text{subject to } Ax \leq b \\ & \quad x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

Proposition 1 Optimal solution to relaxed problem gives upper bound on original problem

Proof show that relaxation

multiplier λ_i “punishment”
 If λ_i large \Rightarrow constraint satisfied
 If $\lambda_i = 0 \Rightarrow$ drop constrain

9

Lagrangian relaxation

Lagrange relaxed problem as function of $\lambda \geq 0$

$$\begin{aligned} & \text{maximize } z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ & \text{subject to } Ax \leq b \\ & \quad x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

Natural questions:

- How do we find best λ ?
- How tight is relaxation?

Properties of Lagrange relaxation

10

Geom. interpretation, Lagrangian Relaxation

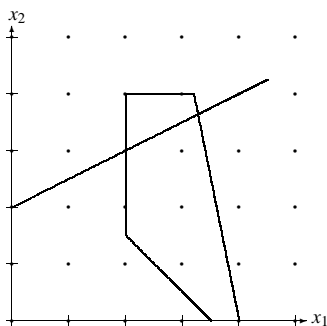
$$\begin{aligned} & \max \quad 7x_1 + 2x_2 \\ & \text{s.t.} \quad -x_1 + 2x_2 \leq 4 \\ & \quad 5x_1 + x_2 \leq 20 \\ & \quad -2x_1 - 2x_2 \leq -7 \\ & \quad -x_1 \leq -2 \\ & \quad x_2 \leq 4 \\ & \quad x_1, x_2 \text{ integer} \end{aligned}$$

First constraint “ $-x_1 + 2x_2 \leq 4$ ” is “complicating”
 Lagrangian relax this constraint ($\lambda \geq 0$) getting objective

$$7x_1 + 2x_2 - \lambda(-x_1 + 2x_2 - 4)$$

Relaxed problem

$$\begin{aligned} & \max \quad (7 + \lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda \\ & \text{s.t.} \quad 5x_1 + x_2 \leq 20 \\ & \quad -2x_1 - 2x_2 \leq -7 \\ & \quad -x_1 \leq -2 \\ & \quad x_2 \leq 4 \\ & \quad x_1, x_2 \text{ integer} \end{aligned}$$



11

Geom. interpretation, Lagrangian Relaxation

Original problem, integer solution

$$(x_1, x_2) = (4, 0) \quad z = 28.00$$

Original problem, LP-relaxed solution

$$(x_1, x_2) = \left(\frac{36}{11}, \frac{40}{11}\right) = (3.27, 3.64) \quad z = 30.18$$

Drop first constraint, integer solution

$$(x_1, x_2) = (3, 4) \quad z = 29.00$$

Drop first constraint, LP-relaxed solution

$$(x_1, x_2) = \left(\frac{16}{5}, 4\right) = (3.2, 4) \quad z = 30.40$$

Maximum on Q , LP-relaxed solution

$$(x_1, x_2) = (3, 4) \quad z = 29.00$$

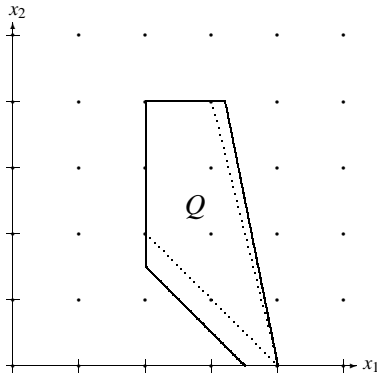
Maximum on Q , with first constraint added

$$(x_1, x_2) = \left(\frac{28}{9}, \frac{32}{9}\right) = (3.11, 3.56) \quad z = 28.88$$

12

Geom. interpretation, Lagrangian Relaxation

Viewpoint 1: fixed λ



$$\begin{aligned} \max \quad & (7+\lambda)x_1 + (2-2\lambda)x_2 + 4\lambda \\ \text{s.t.} \quad & 5x_1 + x_2 \leq 20 \\ & -2x_1 - 2x_2 \leq -7 \\ & -x_1 \leq -2 \\ & x_2 \leq 4 \\ & x_1, x_2 \text{ integer} \end{aligned}$$

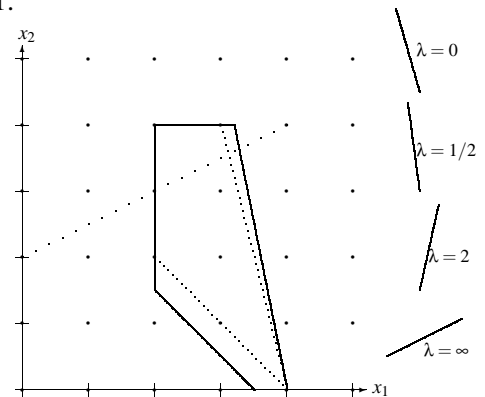
Redefinition using convex hull of Q

$$\begin{aligned} \max \quad & (7+\lambda)x_1 + (2-2\lambda)x_2 + 4\lambda \\ \text{s.t.} \quad & \left. \begin{aligned} 4x_1 + x_2 &\leq 16 \\ -x_1 - x_2 &\leq -4 \\ -x_1 &\leq -2 \\ x_2 &\leq 4 \end{aligned} \right\} Q \end{aligned}$$

13

Geom. interpretation, Lagrangian Relaxation

Viewpoint 1:



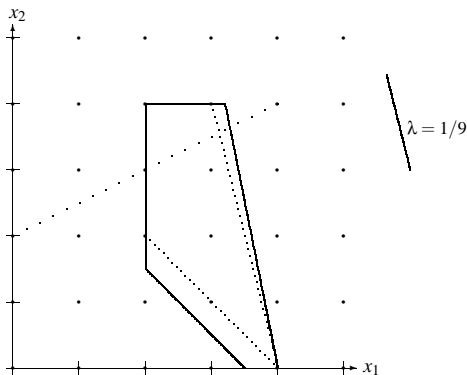
- λ is a modifier of the objective function
- For $0 \leq \lambda \leq \frac{1}{9}$, optimal solution $(3, 4)$

$$z_{LR}(\lambda) = (7+\lambda)3 + (2-2\lambda)4 + 4\lambda = 29 - \lambda$$
- For $\lambda \geq \frac{1}{9}$ optimal solution $(4, 0)$

$$z_{LR}(\lambda) = (7+\lambda)4 + (2-2\lambda)0 + 4\lambda = 28 + 8\lambda$$
- Increasing lambda is forcing the optimal solution to satisfy relaxed constraint.

14

Geom. interpretation, Lagrangian Relaxation



- When $\lambda = \frac{1}{9}$ we get the tightest bound.
- In this case the isoprofit line is parallel to the line through $(3, 4)$ and $(4, 0)$.
- We may choose an arbitrary point x^* on this line

$$(x_1^*, x_2^*) = \left(\frac{28}{9}, \frac{32}{9}\right) = (3.11, 3.56)$$

which satisfies the relaxed constraint

$$-x_1 + 2x_2 \leq 4$$

- In this case

$$z_{LD} = \max \{cx : Dx \leq d, x \in \text{conv}(Q)\}$$

This “proves” theorem 10.3 page 172.

15

Geom. interpretation, Lagrangian Relaxation

Integer Programming Problem

$$\begin{aligned} \text{maximize} \quad & cx \\ \text{subject to} \quad & Ax \leq b \\ & Dx \leq d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

$$\max \left\{ cx : x \in \text{conv}(Ax \leq b, Dx \leq d, x \in \mathbb{Z}_+) \right\}$$

Lagrange Relaxation, multipliers $\lambda \geq 0$

$$\begin{aligned} \text{maximize} \quad & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} \quad & Ax \leq b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

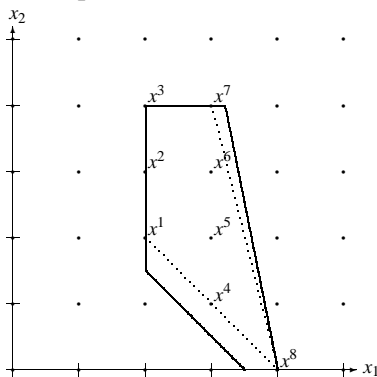
for best multiplier $\lambda \geq 0$

$$\max \left\{ cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+) \right\}$$

16

Geom. interpretation, Lagrangian Relaxation

Viewpoint 2: fixed point x^i



There are 8 integer points in Q :

$$\{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8\} = \{(2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 0)\}$$

For fixed x^i the objective function

$$z_{LR}(\lambda, x^i) = (7 + \lambda)x_1^i + (2 - 2\lambda)x_2^i + 4\lambda = 7x_1^i + 2x_2^i + \lambda(x_1^i - 2x_2^i + 4)$$

is an affine function.

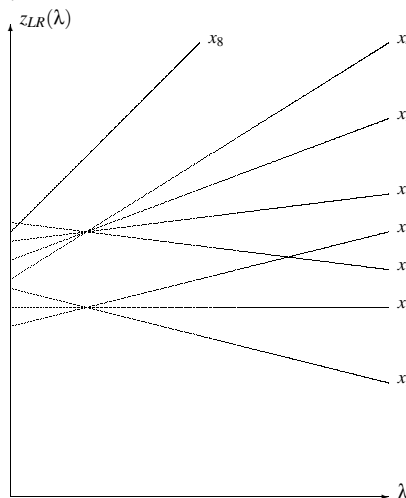
E.g. for $x^7 = (3, 4)$

$$z_{LR}(\lambda, x^7) = 7 \cdot 3 + 2 \cdot 4 + \lambda(3 - 2 \cdot 4 + 4) = 29 - \lambda$$

17

Geom. interpretation, Lagrangian Relaxation

Viewpoint 2:



Objective

$$z_{LR}(\lambda) = \max_{x^i \in Q} z(\lambda, x^i)$$

Proposition 2 The lagrangian relaxed problem $z_{LR}(\lambda)$ as function of the multipliers $\lambda \in \mathbb{R}$, $\lambda \geq 0$ is piecewise linear and convex

(see Wolsey, figure page 173)

18

Lagrangian relaxation and duality

- Lagrangian relaxation is a generalization of duality, where we may “dualize” any subset of constraints.

- Lagrange Relaxation

$$\begin{aligned} &\text{maximize } z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ &\text{subject to } Ax \leq b \\ &\quad x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

is an LP-problem

- Optimal multipliers λ may be found by simplex.
- Subgradient is however faster when few iterations.

19

Lagrangian Relaxation

Integer Programming Problem

$$\begin{aligned} &\text{maximize } cx \\ &\text{subject to } Ax \leq b \\ &\quad Dx \leq d \\ &\quad x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

Lagrange Relaxation, multipliers $\lambda \geq 0$

$$\begin{aligned} &\text{maximize } z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ &\text{subject to } Ax \leq b \\ &\quad x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

Assume that the “nice constraints” $Ax \leq b$ define the convex hull, e.g.

- A is totally unimodular, and b is a vector of integers
- There are no constraints left
- The remaining constraints are defined in linear variables

20

Lagrangian Relaxation

for best multiplier $\lambda \geq 0$ strength of model

$$\max \left\{ cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+) \right\}$$

If $\{x : Ax \leq b\} = \{x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+)\}$ strength

$$\max \left\{ cx : Dx \leq d, Ax \leq b \right\}$$

Corollary (page 173 in Wolsey)

$$z_{LD} = z_{LP}$$

for any objective function cx .

- We do not obtain better bounds than by linear relaxation.
- We may find $z_{LP} = z_{LD}$ in polynomial time.
- If the remaining problem $Ax \leq b$ has a nice structure (e.g. min-spanning-tree) we may find z_{LD} faster than z_{LP} .

21

Lagrangian Relaxation

Lagrangian dual when $Ax \leq b$ define convex hull

$$\begin{aligned} &\text{maximize } cx \\ &\text{subject to } Ax \leq b \\ &\quad Dx \leq d \\ &\quad x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

Consider LP-relaxation with solution z_{LP}

$$\begin{aligned} &\text{maximize } cx \\ &\text{subject to } Ax \leq b \\ &\quad Dx \leq d \\ &\quad x \geq 0 \end{aligned}$$

where the dual problem is

$$\begin{aligned} &\text{minimize } by + dy' \\ &\text{subject to } yA + y'D \geq c \\ &\quad y, y' \geq 0 \end{aligned}$$

Now consider the lagrangian relaxed problem $z_{LR}(\lambda)$

$$\begin{aligned} &\lambda d + \text{maximize } (c - \lambda D)x \\ &\text{subject to } Ax \leq b \\ &\quad x \geq 0 \quad (x \in \mathbb{Z}_+ \text{ for free}) \end{aligned}$$

where the dual problem is

$$\begin{aligned} &\lambda d + \text{minimize } by \\ &\text{subject to } yA \geq c - \lambda D \\ &\quad \lambda, y \geq 0 \end{aligned}$$

22

Lagrangian Relaxation

Integer Programming Problem

$$\begin{aligned} &\text{maximize } cx \\ &\text{subject to } Ax \leq b \\ &\quad Dx \leq d \\ &\quad x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

If $Ax \leq b$ define convex hull, solution to Lagrangian dual

$$\lambda = y'$$

Lagrangian relaxation

- If relax all constraints: ordinary dual problem
- Lagrangian relaxation of a constraint can be seen as “dualization” of a constraint.
- We have found a technique for deriving the best lagrangian multipliers in some special cases.

23