Friday, November 12

Continue from last time:

- Cutting planes — a method to obtain tighter bounds and faster convergence to integer solutions (Wolsey chap. 8)
- Start on Wolsey Chapter 9

Cuts and facets

Definitions

- cuts: valid inequalities
- facets: inequalities defining convex hull

Cuts and facets are redundant for IP formulation
Tighten formulation for LP relaxation

Overview of cuts

- Chvatal cuts
- Gomory cuts (Modular cuts)
- Chvatal-Gomory cuts
- Disjunctive cuts
- Cover inequalities
- Clueq inequalities
- Problem specific cuts

Chvatal Cuts

maximize \( \sum_{j=1}^{n} c_j x_j \)
subject to \( \sum_{j=1}^{n} a_{1j} x_j \leq b_1 \)
\( \vdots \)
\( \sum_{j=1}^{n} a_{mj} x_j \leq b_m \)
\( x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \)

1 Take a linear combination of the constraints
\( \sum_{j=1}^{n} (\sum_{i=1}^{m} u_i a_{ij}) x_j \leq (\sum_{i=1}^{m} u_i b_i) \)
in short \( \sum_{j=1}^{n} a'_{j} x_j \leq b' \)

2 Divide through by a common factor \( d | a'_{j}, j = 1, \ldots, n \)
\( \sum_{j=1}^{n} \frac{a'_{j}}{d} x_j \leq \frac{b'}{d} \)

3 Since all \( a'_{j}/d \) are integers round down \( b' \)
\( \sum_{j=1}^{n} \frac{a'_{j}}{d} x_j \leq \left\lfloor \frac{b'}{d} \right\rfloor \)
**Chvatal-Gomory cuts (p. 119)**

maximize \( \sum_{j=1}^{n} c_j x_j \)

subject to \( \sum_{j=1}^{n} a_{1j} x_j \leq b_1 \)

\[ \vdots \]

\( \sum_{j=1}^{n} a_{mj} x_j \leq b_m \)

\( x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \)

1. Take a linear combination of the constraints 
\( \sum_{j=1}^{n} (\sum_{i=1}^{m} u_{ij}) x_j \leq (\sum_{i=1}^{m} u_i b_i) \)

in short 
\[ \sum_{j=1}^{n} a'_j x_j \leq b' \]

2. Since \( x \geq 0 \) implies \( \sum_{j=1}^{n} (a'_j - |a'_j|) x_j \geq 0 \) we have 
\[ \sum_{j=1}^{n} |a'_j| x_j \leq b' \]

3. Since \( x_j \in \mathbb{Z}_+ \) implies \( |a'_j| x_j \in \mathbb{Z} \) we get 
\[ \sum_{j=1}^{n} |a'_j| x_j \leq \lfloor b' \rfloor \]

**Gomory Cuts**

- Systematical way of generating valid inequalities
- In each step the current LP-solution will be separated
- Ensures that an integer solution will be reached after a number of steps

**Example**

maximize \( 4x_1 - x_2 \)

subject to \( 7x_1 - 2x_2 \leq 14 \)
\( x_1, x_2 \geq 0, \text{integer} \)

The optimal LP-solution is 
\( (x_1, x_2, x_3, x_4, x_5) = (\frac{20}{7}, 3, 0, \frac{23}{7}) \)

which is fractional.

From first equation in Simplex table we get 
\[ x_1 + \frac{7}{2} x_3 + \frac{2}{7} x_4 = \frac{20}{7} \]

and hence also \( x_1 + \frac{7}{2} x_3 + \frac{2}{7} x_4 \leq \frac{20}{7} \), so 
\[ x_1 + \frac{1}{7} x_3 + \frac{2}{7} x_4 \leq \lfloor \frac{20}{7} \rfloor \]

inserting \( x_1 = -\frac{1}{7} x_3 - \frac{2}{7} x_4 + \frac{20}{7} \) we get 
\[ \frac{1}{7} x_3 + \frac{2}{7} x_4 \geq 6 \]

or substituting the slack variables \( x_3 \) and \( x_4 \) we get 
\[ \frac{1}{7}(14 - 7x_1 + 2x_2) + \frac{2}{7}(3 - x_2) \geq 6 \]

which can be reduced to \( x_1 \leq 2 \).
Gomory Cuts

Gomory (1963) presented a general technique for solving IP problems

1. Solve the LP-relaxation
2. Choose one of the basis integer variables taking a fractional value
   \[ x_i + \sum_{j \in \mathbb{N}} a_j x_j = a_0 \]  
   (1)
3. Use the corresponding equation to separate the inequality
   \[ \sum_{j \in \mathbb{N}} (a_j - \lfloor a_j \rfloor) x_j \geq (a_0 - \lfloor a_0 \rfloor) \]  
   (2)
4. Incorporate the new constraint and repeat.

**Proposition 1** Inequality (2) is a valid inequality which separates the current LP solution from the feasible set.

**Proof**

\[ x_i + \sum_{j \in \mathbb{N}} a_j x_j = a_0 \]
\[ \sum_{j \in \mathbb{N}} (a_j - \lfloor a_j \rfloor) x_j \geq (a_0 - \lfloor a_0 \rfloor) \]  
(2)

- [The inequality (2) is valid]
  Since (1) is valid we also have
  \[ x_i + \sum_{j \in \mathbb{N}} a_j x_j \leq a_0 \]
  Derive a C-G cut:
  \[ x_i + \sum_{j \in \mathbb{N}} |a_j| x_j \leq |a_0| \]
  substitute \( x_i = -\sum_{j \in \mathbb{N}} a_j x_j + a_0 \) getting:
  \[ -\sum_{j \in \mathbb{N}} a_j x_j + a_0 + \sum_{j \in \mathbb{N}} |a_j| x_j \leq |a_0| \]
  or
  \[ a_0 - |a_0| \leq \sum_{j \in \mathbb{N}} (a_j - \lfloor a_j \rfloor) x_j \]
- [Inequality (2) separates current solution]
  Current solution was \( x_i = a_0 \) and \( x_j = 0, j \in \mathbb{N} \).
  Inserted in equation (2)
  \[ \sum_{j \in \mathbb{N}} (a_j - \lfloor a_j \rfloor) x_j \geq (a_0 - \lfloor a_0 \rfloor) = 0 \]

Lexicographic order

Given a vector \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \)

\[ v \geq 0 \] \( v \) is lex-positive if first \( v_i \neq 0 \) is positive
\[ v = 0 \] \( v \) is lex-zero if \( v_i = 0, i = 1, \ldots, n \)
\[ v \leq 0 \] \( v \) is lex-negative if first \( v_i \neq 0 \) is negative

Given two vectors \( v, w \in \mathbb{R}^n \)

\[ v < w \] \( v \) is lex-less-than \( w \) if \( v - w < 0 \)
\[ v \geq w \] \( v \) is lex-greater-than \( w \) if \( v - w \geq 0 \)

Define lex-min, lex-max obvious way

**Example**

\((0,0,1,0) \geq \lfloor 0,0,0,2 \rfloor \)
\((0,3,1,2) \leq \lfloor 1,2,4,8 \rfloor \)

Lexicographic Anticycling Rule for Simplex

The primal simplex algorithm terminates after a finite number of pivots if

- Entering variable: choose any column \( s \) with \( a_{0s} < 0 \)
- Leaving variable: choose row by
  \[ \text{lex-min} \frac{a_i}{j: a_{ij} > 0} \]
  where \( a_i = (\text{solution}_i, a_{i1}, a_{i2}, \ldots, a_{in}) \)

**Example (maximization)**

\[
\begin{array}{c|cccccc|c}
\text{basis} & z & x_1 & x_2 & x_3 & x_4 & x_5 & \text{solution} \\
\hline
z & 1 & -3 & 1 & 2 & & & 15 \\
x_1 & 1 & 4 & 1 & 3 & & & 1 \\
x_2 & 1 & 1 & 10 & 1 & & & 2 \\
x_3 & 1 & 12 & 1 & 2 & & & 3 \\
\end{array}
\]

Entering variable \( s = 4 \) since \( a_{0s} = -3 \). Leaving variable

\[ \frac{a_{1s}}{a_{1s}} = (\frac{1}{4}, 1, \frac{1}{2}, \frac{1}{2}) \]
\[ \frac{a_{2s}}{a_{2s}} = (2, 1, 10, 1) \]
\[ \frac{a_{3s}}{a_{3s}} = (\frac{1}{4}, 1, \frac{1}{2}, \frac{1}{2}) \]

Choose row 3.
Gomory Cuts

For pure IP-models we have

**Proposition**

If we always
- derive the Gomory cut from the first simplex row in which the basis variable is fractional
- use the lexicographic version of the simplex algorithm
then Gomory’s fractional algorithm will find an integer optimal solution in a finite number of steps

**However**

There is no bound on the number of steps

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Modular Inequalities

Valid inequalities for

\[ S = \left\{ x \in \mathbb{Z}^n_+ : \sum_{j \in \mathbb{N}} a_j x_j = a_0 \right\} \]

Extend \( S \) to all points which satisfy the inequality plus \( kd \), where \( k > 0 \), integer, \( d \geq 1 \), integer.

\[ S_d = \left\{ x \in \mathbb{Z}^n_+ : \sum_{j \in \mathbb{N}} a_j x_j = a_0 + kd, \text{ some integer } k \right\} \]

Let \( b_j \) be the remainder when \( a_j \) is divided by \( d \). Thus \( a_j = b_j + \alpha_j d \)
where \( 0 \leq b_j < d \) and \( \alpha_j \) integer. Then

\[ S_d = \left\{ x \in \mathbb{Z}^n_+ : \sum_{j \in \mathbb{N}} b_j x_j = b_0 + kd, \text{ some integer } k \right\} \]

The integer \( k \) must satisfy
\[ k = \frac{\sum_{j \in \mathbb{N}} b_j x_j - b_0}{d} \geq 0 - \frac{b_0}{d} \quad \text{since } \sum_{j \in \mathbb{N}} b_j x_j \geq 0 \]
\[ > -1 \quad \text{since } b_0 / d < 1 \]
\[ \geq 0 \quad \text{since } k \text{ integer} \]

Thus we have the valid inequality
\[ \sum_{j \in \mathbb{N}} b_j x_j \geq b_0 \]

Since \( S \subset S_d \), inequality is valid for \( S \).

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Disjunctive Inequalities (section 8.8)

**Proposition** Assume that

\[ \sum_{j \in \mathbb{N}} \pi_j x_j \leq \pi_0 \]

is a valid inequality for \( X_1 \) and

\[ \sum_{j \in \mathbb{N}} \pi'_j x_j \leq \pi'_0 \]

is a valid inequality for \( X_2 \). Then

\[ \sum_{j \in \mathbb{N}} \min(\pi_j, \pi'_j) x_j \leq \max(\pi_0, \pi'_0) \]

is a valid inequality for \( X_1 \cup X_2 \).

**Proof** If we have the valid inequality

\[ \sum_{j \in \mathbb{N}} \pi_j x_j \leq \pi_0 \]

then also

\[ \sum_{j \in \mathbb{N}} b_j x_j \leq b_0 \]

is a valid inequality if \( b_j \leq \pi_j \) and \( b_0 \geq \pi_0 \). \( \square \)

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Example

Valid inequality for \( X_1 \)

\[ 2x_1 + 3x_2 + 7x_3 \leq 15 \]

Valid inequality for \( X_2 \)

\[ 4x_1 + 2x_2 + 9x_3 \leq 11 \]

Then valid inequality for \( X_1 \cup X_2 \) is

\[ 2x_1 + 2x_2 + 7x_3 \leq 15 \]
Wolsey chapter 9

Deeper understanding of cuts, facets

- We would like to use the “best” formulation
- Dominance, redundance, facets
- Facets are intuitively easy to understand
- How to prove that a valid inequality is facet defining?

Notice

\[ \pi x \leq \pi_0 \]

and

\[ \lambda \pi x \leq \lambda \pi_0 \]

are identical for any \( \lambda > 0 \)

Dominance

\[
\begin{align*}
\text{maximize} & \quad \ldots \\
\text{subject to} & \quad 1x_1 + 3x_2 \leq 4 \\
 & \quad 2x_1 + 4x_2 \leq 9 \\
 & \quad x_1, x_2 \geq 0
\end{align*}
\]

Multiplying the second inequality with \( u = \frac{1}{2} \)

\[ 1x_1 + 2x_2 \leq \frac{9}{2} \]

First inequality dominates the second.

Dominance:

\[ \pi x \leq \pi_0 \quad \mu x \leq \mu_0 \]

\( \pi x \leq \pi_0 \) dominates \( \mu x \leq \mu_0 \) if there exists \( u > 0 \) such that \( \pi \geq u\mu \) and \( \pi_0 \leq u\mu_0 \).

Redundance

\[
\begin{align*}
\text{maximize} & \quad \ldots \\
\text{subject to} & \quad 6x_1 - x_2 \leq 9 \\
 & \quad 9x_1 - 5x_2 \leq 6 \\
 & \quad 5x_1 - 2x_2 \leq 6 \\
 & \quad x_1, x_2 \geq 0
\end{align*}
\]

Multiplying the first two constraints with \( u = \left( \frac{1}{3}, \frac{1}{4} \right) \)

\[ 5x_1 - 2x_2 \leq 5 \]

Last inequality is redundant

Polyhedra, Facets

Polyhedra \( P \subset \mathbb{R}^2 \)

\[
\begin{align*}
\text{subject to} & \quad x_1 + x_2 \leq 2 \\
 & \quad x_1 + 2x_2 \leq 4 \\
 & \quad x_1 + 2x_2 \leq 10 \\
 & \quad x_1 + x_2 \geq 6 \\
 & \quad x_1, x_2 \geq 2 \\
 & \quad x_1, x_2 \geq 0
\end{align*}
\]

- \( P \subset \mathbb{R}^2 \) and “both directions are present”
- \( P \) is full-dimensional.
- The points \( (2, 0), (1, 1) \) and \( (2, 2) \) are affinely independent points.
- The vectors \( (2, 0, 1), (1, 1, 1) \) and \( (2, 2, 1) \) are linearly independent.
- The dimension of \( P \) is one less than the number of affinely independent points.
Polyhedra, Facets

Affinely independent

The points $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ are affinely independent if the $k-1$ directions $(x^2 - x^1), \ldots, (x^k - x^1)$ are linearly independent.

Dimension

The dimension of $P$, denoted $\dim(P)$, is one less than the maximum number of affinely independent points in $P$.

A line is 1-dim
A plane is 2-dim
A box is 3-dim

Full-dimensional

The polyhedra $P \subseteq \mathbb{R}^n$ is full-dimensional if and only if $\dim(P) = n$.

Face, Facets

If $\pi x \leq \pi_0$ is a valid inequality of $P$ then

$$F = \{x \in P : \pi x = \pi_0\}$$

defines a face of $P$.

$F$ is a facet of $P$ iff

- $F$ is a face of $P$
- $\dim(F) = \dim(P) - 1$

Minimal description of previous example

subject to $x_1 \leq 2$

$x_1 + x_2 \leq 4$

$x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$

Polyhedra, Facets

subject to $x_1 \leq 2$

$x_1 + x_2 \leq 4$

$x_1 + 2x_2 \leq 10$

$x_1 + x_2 \leq 6$

$x_1, x_2 \geq 2$

$x_1, x_2 \geq 0$

IP-problems

Two-dimensional problem

$$P = \{x : Ax \leq b\} \cap \mathbb{Z}^2$$

is dotted line a facet?

- Dimension of $P$ is 2
- The facet defining inequality must be valid
- A facet should have dimension 1
- There should be 2 affinely independent points on a facet