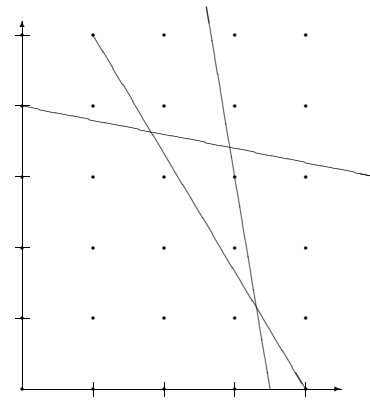


Continue from last time:

- Cutting planes — a method to obtain tighter bounds and faster convergence to integer solutions (Wolsey chap. 8)
- Start on Wolsey Chapter 9



Definitions

- cuts: valid inequalities
- facets: inequalities defining convex hull

Cuts and facets are redundant for IP formulation  
Tighten formulation for LP relaxation

**Overview of cuts**

- Chvatal cuts
- Gomory cuts (Modular cuts)
- Chvatal-Gomory cuts
- Disjunctive cuts
- Cover inequalities
- Clique inequalities
- Problem specific cuts

**Chvátal Cuts**

$$\begin{aligned} &\text{maximize } \sum_{j=1}^n c_j x_j \\ &\text{subject to } \sum_{j=1}^n a_{1j} x_j \leq b_1 \\ &\quad \vdots \\ &\quad \sum_{j=1}^n a_{mj} x_j \leq b_m \\ &\quad x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

1 Take a linear combination of the constraints

$$\sum_{j=1}^n \left( \sum_{i=1}^m u_i a_{ij} \right) x_j \leq \left( \sum_{i=1}^m u_i b_i \right)$$

in short

$$\sum_{j=1}^n a'_j x_j \leq b'$$

2 Divide through by a common factor  $d|a'_j, j = 1, \dots, n$

$$\sum_{j=1}^n \frac{a'_j}{d} x_j \leq \frac{b'}{d}$$

3 Since all  $a'_j/d$  are integers round down  $b'$

$$\sum_{j=1}^n \frac{a'_j}{d} x_j \leq \lfloor \frac{b'}{d} \rfloor$$



## Gomory Cuts

Gomory (1963) presented a general technique for solving IP problems

- 1 Solve the LP-relaxation
- 2 Choose one of the basis integer variables taking a fractional value

$$x_i + \sum_{j \in N} a_j x_j = a_0 \quad (1)$$

- 3 Use the corresponding equation to separate the inequality

$$\sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j \geq (a_0 - \lfloor a_0 \rfloor) \quad (2)$$

- 4 Incorporate the new constraint and repeat.

**Proposition 1** Inequality (2) is a valid inequality which separates the current LP solution from the feasible set.

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## Proof

$$x_i + \sum_{j \in N} a_j x_j = a_0 \quad (1)$$

$$\sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j \geq (a_0 - \lfloor a_0 \rfloor) \quad (2)$$

- [The inequality (2) is valid]

Since (1) is valid we also have

$$x_i + \sum_{j \in N} a_j x_j \leq a_0$$

Derive a C-G cut

$$x_i + \sum_{j \in N} \lfloor a_j \rfloor x_j \leq \lfloor a_0 \rfloor$$

substitute  $x_i = -\sum_{j \in N} a_j x_j + a_0$  getting

$$-\sum_{j \in N} a_j x_j + a_0 + \sum_{j \in N} \lfloor a_j \rfloor x_j \leq \lfloor a_0 \rfloor$$

or

$$a_0 - \lfloor a_0 \rfloor \leq \sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j$$

- [Inequality (2) separates current solution]

Current solution was  $x_i = a_0$  and  $x_j = 0, j \in N$ .  
Inserted in equation (2)

$$\sum_{j \in N} (a_j - \lfloor a_j \rfloor) 0 \geq (a_0 - \lfloor a_0 \rfloor) > 0$$

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## Lexicographic order

Given a vector  $v = (v_1, \dots, v_n) \in \mathbb{R}^n$

$v \underset{L}{>} 0$   $v$  is lex-positive if first  $v_i \neq 0$  is positive

$v \underset{L}{=} 0$   $v$  is lex-zero if  $v_i = 0, i = 1, \dots, n$

$v \underset{L}{<} 0$   $v$  is lex-negative if first  $v_i \neq 0$  is negative

Given two vectors  $v, w \in \mathbb{R}^n$

$v \underset{L}{<} w$   $v$  is lex-less-than  $w$  if  $v - w \underset{L}{<} 0$

$v \underset{L}{>} w$   $v$  is lex-greater-than  $w$  if  $v - w \underset{L}{>} 0$

Define lex-min, lex-max obvious way

## Example

$$(0, 0, 1, 0) \underset{L}{>} (0, 0, 0, 2)$$

$$(0, 3, 1, 2) \underset{L}{<} (1, 2, 4, 8)$$

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## Lexicographic Anticycling Rule for Simplex

The primal simplex algorithm terminates after a finite number of pivots if

- Entering variable: choose any column  $s$  with  $a_{0s} < 0$
- Leaving variable: choose row by

$$\text{lex-min}_{i: a_{is} > 0} \frac{a_i}{a_{is}}$$

where  $a_i = (\text{solution}_i, a_{i1}, a_{i2}, \dots, a_{in})$

## Example (maximization)

basis	$z$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	solution
$z$	1				-3	1	2	15
$x_1$		1			4	1	3	1
$x_2$			1		1	10	1	2
$x_3$				1	12	1	2	3

Entering variable  $s = 4$  since  $a_{0s} = -3$ . Leaving variable

$$\frac{a_1}{a_{1s}} = \left(\frac{1}{4}, 1, \frac{1}{4}, \frac{3}{4}\right) \quad \frac{a_2}{a_{2s}} = (2, 1, 10, 1) \quad \frac{a_3}{a_{3s}} = \left(\frac{1}{4}, 1, \frac{1}{12}, \frac{1}{6}\right)$$

Choose row 3.

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## Gomory Cuts

For pure IP-models we have

### Proposition

If we always

- derive the Gomory cut from the first simplex row in which the basis variable is fractional
- use the lexicographic version of the simplex algorithm

then Gomory's fractional algorithm will find an integer optimal solution in a finite number of steps

### However

There is no bound on the number of steps

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## Modular Inequalities

Valid inequalities for

$$S = \left\{ x \in \mathbb{Z}_+^n : \sum_{j \in N} a_j x_j = a_0 \right\}$$

Extend  $S$  to all points which satisfy the inequality plus  $kd$ , where  $k > 0$ , integer,  $d \geq 1$ , integer.

$$S_d = \left\{ x \in \mathbb{Z}_+^n : \sum_{j \in N} a_j x_j = a_0 + kd, \text{ some integer } k \right\}$$

Let  $b_j$  be the remainder when  $a_j$  is divided by  $d$ . Thus

$$a_j = b_j + \alpha_j d$$

where  $0 \leq b_j < d$  and  $\alpha_j$  integer. Then

$$S_d = \left\{ x \in \mathbb{Z}_+^n : \sum_{j \in N} b_j x_j = b_0 + kd, \text{ some integer } k \right\}$$

The integer  $k$  must satisfy

$$\begin{aligned} k &= \frac{\sum b_j x_j}{d} - \frac{b_0}{d} \\ &\geq 0 - \frac{b_0}{d} && \text{since } \sum_{j \in N} b_j x_j \geq 0 \\ &> -1 && \text{since } b_0/d < 1 \\ &\geq 0 && \text{since } k \text{ integer} \end{aligned}$$

Thus we have the valid inequality

$$\sum_{j \in N} b_j x_j \geq b_0$$

Since  $S \subset S_d$ , inequality is valid for  $S$ .

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## Disjunctive Inequalities (section 8.8)

**Proposition** Assume that

$$\sum_{j \in N} \pi_j x_j \leq \pi_0$$

is a valid inequality for  $X_1$  and

$$\sum_{j \in N} \pi'_j x_j \leq \pi'_0$$

is a valid inequality for  $X_2$ . Then

$$\sum_{j \in N} \min(\pi_j, \pi'_j) x_j \leq \max(\pi_0, \pi'_0)$$

is a valid inequality for  $X_1 \cup X_2$ .

**Proof** If we have the valid inequality

$$\sum_{j \in N} \pi_j x_j \leq \pi_0$$

then also

$$\sum_{j \in N} b_j x_j \leq b_0$$

is a valid inequality if  $b_j \leq \pi_j$  and  $b_0 \geq \pi_0$ .  $\square$

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## Example

Valid inequality for  $X_1$

$$2x_1 + 3x_2 + 7x_3 \leq 15$$

Valid inequality for  $X_2$

$$4x_1 + 2x_2 + 9x_3 \leq 11$$

Then valid inequality for  $X_1 \cup X_2$  is

$$2x_1 + 2x_2 + 7x_3 \leq 15$$

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Deeper understanding of cuts, facets

- We would like to use the “best” formulation
- Dominance, redundancy, facets
- Facets are intuitively easy to understand
- How to prove that a valid inequality is facet defining?

Notice

$$\pi x \leq \pi_0$$

and

$$\lambda \pi x \leq \lambda \pi_0$$

are identical for any  $\lambda > 0$

Dominance

$$\begin{aligned} &\text{maximize } \dots \\ &\text{subject to } 1x_1 + 3x_2 \leq 4 \\ &\quad \quad \quad 2x_1 + 4x_2 \leq 9 \\ &\quad \quad \quad x_1, x_2 \geq 0 \end{aligned}$$

Multiplying the second inequality with  $u = \frac{1}{2}$

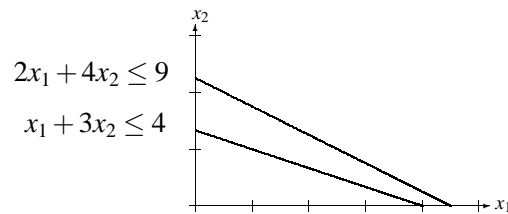
$$1x_1 + 2x_2 \leq \frac{9}{2}$$

First inequality dominates the second.

Dominance:

$$\pi x \leq \pi_0 \quad \mu x \leq \mu_0$$

$\pi x \leq \pi_0$  dominates  $\mu x \leq \mu_0$  if there exists  $u > 0$  such that  $\pi \geq u\mu$  and  $\pi_0 \leq u\mu_0$ .



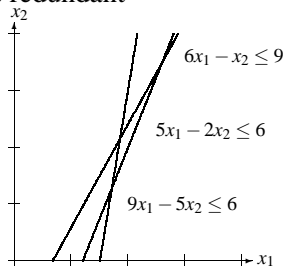
Redundance

$$\begin{aligned} &\text{maximize } \dots \\ &\text{subject to } 6x_1 - x_2 \leq 9 \\ &\quad \quad \quad 9x_1 - 5x_2 \leq 6 \\ &\quad \quad \quad 5x_1 - 2x_2 \leq 6 \\ &\quad \quad \quad x_1, x_2 \geq 0 \end{aligned}$$

Multiplying the first two constraints with  $u = (\frac{1}{3}, \frac{1}{3})$

$$5x_1 - 2x_2 \leq 5$$

Last inequality is redundant



Redundance:

$$\pi^i x \leq \pi_0^i, \quad i = 1, \dots, k$$

$$\mu x \leq \mu_0$$

Inequality  $\mu x \leq \mu_0$  is *redundant* if there exists a vector  $(u_1, \dots, u_k) \geq 0$  such that

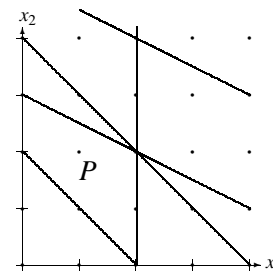
$$\left( \sum_{i=1}^k u_i \pi^i \right) x \leq \sum_{i=1}^k u_i \pi_0^i$$

dominates  $\mu x \leq \mu_0$

Polyhedra, Facets

Polyhedra  $P \subset \mathbb{R}^2$

$$\begin{aligned} &\text{subject to } x_1 \leq 2 \\ &\quad \quad \quad x_1 + x_2 \leq 4 \\ &\quad \quad \quad x_1 + 2x_2 \leq 10 \\ &\quad \quad \quad x_1 + 2x_2 \leq 6 \\ &\quad \quad \quad x_1 + x_2 \geq 2 \\ &\quad \quad \quad x_1, x_2 \geq 0 \end{aligned}$$



- $P \subset \mathbb{R}^2$  and “both directions are present”
- $P$  is full-dimensional.
- The points  $(2, 0)$ ,  $(1, 1)$  and  $(2, 2)$  are affinely independent points.
- The vectors  $(2, 0, 1)$ ,  $(1, 1, 1)$  and  $(2, 2, 1)$  are linearly independent.
- The dimension of  $P$  is one less than the number of affinely independent points.

## Polyhedra, Facets

### Affinely independant

The points  $x^1, x^2, \dots, x^k \in \mathbb{R}^n$  are affinely independant if the  $k - 1$  directions  $(x^2 - x^1), \dots, (x^k - x^1)$  are linearly independant.

### Dimension

The dimension of  $P$ , denoted  $\dim(P)$  is one less than the maximum number of affinely independant points in  $P$ .

A line is 1-dim  
 A plane is 2-dim  
 A box is 3-dim

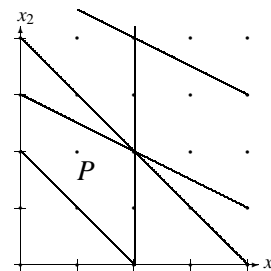
### Full-dimensional

The polyhedra  $P \subseteq \mathbb{R}^n$  is full-dimensional if and only if  $\dim(P) = n$ .

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## Polyhedra, Facets

$$\begin{aligned} \text{subject to } x_1 &\leq 2 \\ x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 10 \\ x_1 + 2x_2 &\leq 6 \\ x_1 + x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$



- $x_1 \leq 2$  defines a facet of  $P$ , as  $(2, 0)$  and  $(2, 2)$  are two affinely independant points in  $P$ .
- $x_1 + 2x_2 \leq 6$  defines a facet.
- $x_1 + x_2 \geq 2$  defines a facet.
- $x_1 \geq 0$  defines a facet.
- $x_1 + x_2 \leq 4$  is a face with one point  $(2, 2) \in P$
- $x_1 + x_2 \leq 4$  is redundant:  $u = (\frac{1}{2}, 0, 0, \frac{1}{2}, 0)$
- $x_2 \geq 0$  is redundant:  $u = (1, 0, 0, 0, -1)$

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## Face, Facets

If  $\pi x \leq \pi_0$  is a valid inequality of  $P$  then

$$F = \{x \in P : \pi x = \pi_0\}$$

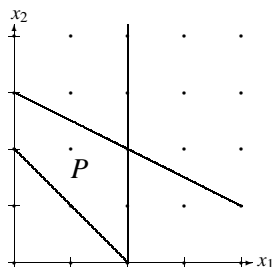
defines a face of  $P$ .

$F$  is a facet of  $P$  iff

- $F$  is a face of  $P$
- $\dim(F) = \dim(P) - 1$

Minimal description of previous example

$$\begin{aligned} \text{subject to } x_1 &\leq 2 \\ x_1 + 2x_2 &\leq 6 \\ x_1 + x_2 &\geq 2 \\ x_1 &\geq 0 \end{aligned}$$



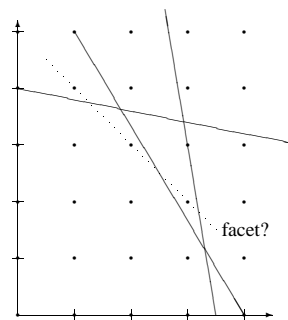
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## IP-problems

Two-dimensional problem

$$P = \{x : Ax \leq b\} \cap \mathbb{Z}^2$$

is dotted line a facet?



- Dimension of  $P$  is 2
- The facet defining inequality must be valid
- A facet should have dimension 1
- There should be 2 affine independant points on a facet

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