Wednesday, November 10

Program of the day:

- Cutting planes — a method to obtain tighter bounds and faster convergence to integer solutions (Wolsey chap. 8)
- Application: branch-and-cut algorithms
Introduction

- branch-and-bound: divide and conquer.
- cutting plane: add inequalities which separate fractional solution from solution space.

Development

- 50’s cutting plane (Gomory: simplex, no $\mathcal{NP}$-hardness)
- 70’s tighten formulation in preprocessing
- 80-90’s branch-and-cut (Padberg, Rinaldi)

Preprocessing $\rightarrow$ part of solution process

Definitions

- cuts: valid inequalities
- facets: inequalities defining convex hull

Cuts and facets are redundant for IP formulation
Tighten formulation for LP relaxation
Cuts and facets
Examples from last lesson

Preprocessing, integer variables

maximize \[ \ldots \]
subject to \[ 7x_1 + 3x_2 - 4x_3 - 2x_4 \leq 1 \]
\[ -2x_1 + 7x_2 + 3x_3 + 4x_4 \leq 6 \]
\[ -2x_2 - 3x_3 - 6x_4 \leq -5 \]
\[ 3x_1 - 2x_3 \geq -1 \]
\[ x \in \mathbb{B}^4 \]

Generating logical inequalities

From constraint 1 we see that

- if \( x_1 = 1 \) and \( x_2 = 1 \) then infeasible, thus
\[ x_1 + x_2 \leq 1 \]
Examples from last lesson

maximize \( x_1 + x_2 \)
subject to
\[-2x_1 + 2x_2 \geq 1\]
\[-8x_1 + 10x_2 \leq 13\]
\( x_1, x_2 \geq 0, \text{ integer} \)

Tightening formulation

\[-2x_1 + 2x_2 \geq 1\]
\[-x_1 + x_2 \geq 1/2\]
\[-x_1 + x_2 \geq 1\]
\[-8x_1 + 10x_2 \leq 13\]
\[-4x_1 + 5x_2 \leq 13/2\]
\[-4x_1 + 5x_2 \leq 6\]
Motivation

Integer programming problem (IP)

\[
\max \{ cx : x \in X \}
\]

where \( X = \{ x : Ax \leq b, x \in \mathbb{Z}_+^n \} \). Reformulate to

\[
\max \{ cx : x \in \text{conv}(X) \}
\]

For any \( c \), an optimal solution to LP is also optimal to IP

Valid inequalities (def. 8.1)

Consider the problem:

\[
\begin{align*}
\text{maximize} & \quad f(x) \\
\text{subject to} & \quad x \in X
\end{align*}
\]

An inequality

\[
\pi x \leq \pi_0
\]

is a valid inequality for \( X \subseteq \mathbb{R}^n \) if

\[
\pi x \leq \pi_0 \quad \text{for all} \quad x \in X
\]
Characterisation of valid inequalities (sec. 8.3.2)

Consider the problem:

\[
\begin{align*}
\text{maximize} & \quad f(x) \\
\text{subject to} & \quad x \in X
\end{align*}
\]

where

\[X = \{y \in \mathbb{Z} : y \leq b\}\]

then the inequality

\[y \leq \lfloor b \rfloor\]

is valid for \(X\)

- Simple observation
- Complete characterisation
Overview of cuts

- Chvatal cuts
- Gomory cuts (Modular cuts)
- Chvatal-Gomory cuts
- Disjunctive cuts
- Cover inequalities
- Clique inequalities
- Problem specific cuts

Notice

- Cuts and facets are independant of objective function
- A tight formulation can be used for any objective
Example of Facets

The problem

\[
\begin{align*}
\text{minimize} & \quad 2x_1 + 7x_2 + 2x_3 \\
\text{subject to} & \quad x_1 + 4x_2 + x_3 \geq 10 \\
& \quad 4x_1 + 2x_2 + 2x_3 \geq 13 \\
& \quad x_1 + x_2 - x_3 \geq 0 \\
& \quad x_1, x_2, x_3 \geq 0, \text{ integer}
\end{align*}
\]

has the facets

\[
\begin{align*}
& \quad x_1 + 4x_2 + x_3 \geq 10 \\
& \quad 2x_1 + x_2 + x_3 \geq 7 \\
& \quad x_1 + x_2 - x_3 \geq 0 \\
& \quad x_1 + 3x_2 + x_3 \geq 9 \\
& \quad 2x_1 + 4x_2 + x_3 \geq 13 \\
& \quad x_1 + x_2 + x_3 \geq 5 \\
& \quad x_1 + 2x_2 \geq 5 \\
& \quad 2x_1 + x_2 \geq 4 \\
& \quad x_1 \geq 0, \text{ integer} \\
& \quad x_2 \geq 0, \text{ integer} \\
& \quad x_3 \geq 0, \text{ integer}
\end{align*}
\]

Using the new formulation we obtain an integer optimal solution by solving the LP-relaxed problem. (For any objective function).
Chvátal Cuts

Valid inequalities for a pure IP-model (minimization)
1 Add constraints, using suitable multipliers
2 Divide through by a common coefficient factor
3 Round up right-hand-side to the next integer

Example

minimize \( 2x_1 + 7x_2 + 2x_3 \)
subject to  
\[
\begin{align*}
    x_1 + 4x_2 + x_3 & \geq 10 \quad (1) \\
    4x_1 + 2x_2 + 2x_3 & \geq 13 \quad (2) \\
    x_1 + x_2 - x_3 & \geq 0 \quad (3) \\
    x_1 & \geq 0 \quad (4) \\
    x_2 & \geq 0 \quad (5) \\
    x_3 & \geq 0 \quad (6)
\end{align*}
\]

\( x_1, x_2, x_3 \) integer

1 times (2) is
\[
4x_1 + 2x_2 + 2x_3 \geq 13
\]

divide by two
\[
2x_1 + x_2 + x_3 \geq 6\frac{1}{2}
\]

left hand side is integral, thus round up right-hand
\[
2x_1 + x_2 + x_3 \geq 7
\]
Example (continued)

minimize \[ 2x_1 + 7x_2 + 2x_3 \]
subject to \[ x_1 + 4x_2 + x_3 \geq 10 \] \hspace{1cm} (1)
\[ 4x_1 + 2x_2 + 2x_3 \geq 13 \] \hspace{1cm} (2)
\[ x_1 + x_2 - x_3 \geq 0 \] \hspace{1cm} (3)
\[ x_1 \geq 0 \] \hspace{1cm} (4)
\[ x_2 \geq 0 \] \hspace{1cm} (5)
\[ x_3 \geq 0 \] \hspace{1cm} (6)
\[ x_1, x_2, x_3 \text{ integer} \]

Facets

\[ x_1 + 4x_2 + x_3 \geq 10 \] \hspace{1cm} (a)
\[ 2x_1 + x_2 + x_3 \geq 7 \] \hspace{1cm} (b)
\[ x_1 + x_2 - x_3 \geq 0 \] \hspace{1cm} (c)
\[ x_1 + 3x_2 + x_3 \geq 9 \] \hspace{1cm} (d)
\[ 2x_1 + 4x_2 + x_3 \geq 13 \] \hspace{1cm} (e)
\[ x_1 + x_2 + x_3 \geq 5 \] \hspace{1cm} (f)
\[ x_1 + 2x_2 \geq 5 \] \hspace{1cm} (g)
\[ 2x_1 + x_2 \geq 4 \] \hspace{1cm} (h)
\[ x_1, x_2, x_3 \geq 0, \text{ integer} \]

Obtained as

(d) : \( 5 \times (1), 1 \times (b), 1 \times (6), \text{ divide 7} \)
(f) : \( 1 \times (1), 3 \times (b), 3 \times (6), \text{ divide 7} \)
(g) : \( 4 \times (1), 1 \times (b), 5 \times (3), \text{ divide 11} \)
(h) : \( 1 \times (b), 1 \times (c), 1 \times (4), \text{ divide 2} \)
(e) : \( 3 \times (d), 1 \times (b), 3 \times (g), \text{ divide 4} \)
Chvatal-Gomory cuts (p. 119)

maximize \( \sum_{j=1}^{n} c_j x_j \)

subject to \( \sum_{j=1}^{n} a_{1j} x_j \leq b_1 \)

\[ \vdots \]

\( \sum_{j=1}^{n} a_{mj} x_j \leq b_m \)

\( x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \)

1. Take a linear combination of the constraints

\( \sum_{j=1}^{n} \left( \sum_{i=1}^{m} u_i a_{ij} \right) x_j \leq \left( \sum_{i=1}^{m} u_i b_i \right) \)

in short

\( \sum_{j=1}^{n} a'_{j} x_j \leq b' \)

2. Since \( x \geq 0 \) implies \( \sum_{j=1}^{n} (a'_{j} - \lfloor a'_{j} \rfloor) x_j \geq 0 \) we have

\( \sum_{j=1}^{n} \lfloor a'_{j} \rfloor x_j \leq b' \)

3. Since \( x_j \in \mathbb{Z}_+ \) implies \( \lfloor a'_{j} \rfloor x_j \in \mathbb{Z} \) we get

\( \sum_{j=1}^{n} \lfloor a'_{j} \rfloor x_j \leq \lfloor b' \rfloor \)
Chvatal-Gomory (Theorem 8.4)

\[
X = \{ x : Ax \leq b, x \in \mathbb{Z}^n_+ \}
\]

Every valid inequality for \(X\) can be obtained by applying the Chvatal-Gomory procedure a finite number of times.

Notice

- No stronger inequalities than Chvatal-Gomory exists.
- Even the facet constraints can be generated as Chvatal-Gomory cuts.
- No constructive (polynomial) algorithm for how the linear combination of constraints should be chosen.
- In practice, the derivation of Chvatal-Gomory cuts must rely on specific features of a given application.

Gomory cuts is a systematical way of deriving cutting planes.

Only 0-1 case

All bounded integer variables can be expressed as sum of binary variables.
The set $X = P \cap \mathbb{Z}^n$

$P = \{x \in \mathbb{R}^n : Ax \leq b, 0 \leq x \leq 1\} \neq \emptyset$
$X = P \cap \mathbb{Z}^n$

Nemhauser and Wolsey, Proposition 1.1 page 208:

If inequality $\pi x \leq \pi_0$ is valid for $P$ then it can be obtained as a C-G cut

(*)

- LP-redundant constraints are C-G inequalities
- Theorem 8.4 deals with IP-constraints
Proof (0-1 case)

Assume that
\[ \pi x \leq \pi_0 \quad \text{where} \quad \pi, \pi_0 \text{ integers} \]
is a valid inequality for \( X \). We will show that this inequality can be obtained by using the C-G procedure a finite number of times.

- Step 1: Find a large number \( t \in \mathbb{Z}_+ \) such that
  \[ \pi x \leq \pi_0 + t \]
is a valid C-G inequality

- Step 2: Prove that if
  \[ \pi x \leq \pi_0 + \tau + 1 \]
for \( \tau \in \mathbb{Z}_+ \) is a C-G inequality for \( X \) then also
  \[ \pi x \leq \pi_0 + \tau \]
is a C-G inequality for \( X \).

- Step 3: Use step 2 for \( \tau = t - 1, \ldots, 0 \) each time getting a new C-G inequality

(Proof by induction)
Step 1

The inequality \[ \pi x \leq \pi_0 + t \]
is valid for \( P \) for some \( t \in \mathbb{Z}_+ \).

Proof

We have the inequality

\[ x \leq 1 \]
derive C-G inequality using multipliers \( u = \pi \)

\[ \pi x \leq \pi 1 \]

choosing \( t = \pi 1 - \pi_0 \) (\( \pi, \pi_0 \) is integer) we get the form

\[ \pi x \leq \pi 1 = \pi_0 + t \]

for some \( t \in \mathbb{Z}_+ \)

Note that \( t < \infty \) as \( P \subseteq [0, 1]^n \) is bounded so \( \max\{\pi x \mid x \in P\} < \infty \).
**Step 2**

Difficult part

a) Prove that if \( \pi x \leq \pi_0 + \tau + 1 \) with \( \tau \in \mathbb{Z}_+ \) is a C-G cut then
\[
\pi x \leq \pi_0 + \tau + \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)
\]
is a C-G inequality for \( X \) for every partition \((N^0, N^1)\) of \( N = \{1, \ldots, n\} \).

b) Use partitionings \((T^0 \cup \{n\}, T^1)\) and \((T^0, T^1 \cup \{n\})\) to obtain a new inequality for \((T^0, T^1)\).

c) Derive all valid inequalities for partitionings of \( N' = \{1, \ldots, n - 1\} \)

d) Repeating this procedure \( n \) times implies that we eliminate the sums on the right side and thus
\[
\pi x \leq \pi_0 + \tau
\]
is a C-G cut

**Time complexity**

- part (c) takes \( O(2^n) \),
- part (d) is performed \( n \) times, in total \( O(n2^n) \)
- we run Step 2 \( O(t) \) times, thus in total \( O(tn2^n) \).
Step 2, a)

An inequality is valid for $P$ if it is valid for all extreme points $\{x^1, \ldots, x^m\}$ of $P$.

Assume that $\pi x \leq \pi_0 + \tau + 1$ with $\tau \in \mathbb{Z}_+$ is a valid cut. Let $(N^0, N^1)$ be any partitioning of $N = \{1, \ldots, n\}$. Consider an extreme point $x^k$ of $P$.

- $x^k$ integer: then $\pi x^k \leq \pi_0$ (since $\pi x \leq \pi_0$ valid for $X$)
- $x^k$ fractional: exists $\varepsilon > 0$ such that

$$\varepsilon^k \leq \sum_{j \in N^0} x^k_j + \sum_{j \in N^1} (1 - x^k_j)$$

Choose $\alpha = \min_{x^k \text{ vertex in } P} \varepsilon^k$.

Using $M \geq (\tau + 1)/\alpha$, we have

$$\tau + 1 \leq M\alpha \leq M \left( \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j) \right)$$

adding $\pi_0$ at both sides we get valid inequality for $P$.

$$\pi x \leq \pi_0 + \tau + 1 \leq \pi_0 + M \left( \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j) \right)$$

Due to (*) the inequality is a C-G cut.
Step 2, a)

We have just shown that the following is a C-G inequality

\[ \pi x \leq \pi_0 + M \left( \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j) \right) \]

By assumption we had the C-G inequality

\[ \pi x \leq \pi_0 + \tau + 1 \]

use weights \(1/M\) and \((M - 1)/M\) for the two inequalities getting C-G inequality

\[ \pi x \leq \pi_0 + \tau + \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j) \]
Step 2, b)

Use partitions \((T^0 \cup \{n\}, T^1)\) and \((T^0, T^1 \cup \{n\})\)

\[ \pi x \leq \pi_0 + \tau + \sum_{j \in T^0 \cup \{n\}} x_j + \sum_{j \in T^1} (1 - x_j) \]

and

\[ \pi x \leq \pi_0 + \tau + \sum_{j \in T^0} x_j + \sum_{j \in T^1 \cup \{n\}} (1 - x_j) \]

using multipliers \(1/2\) and \(1/2\) we get C-G inequality

\[ \pi x \leq \pi_0 + \tau + \sum_{j \in T^0} x_j + \sum_{j \in T^1} (1 - x_j) \]
Branch-and-cut algorithms

Combines best properties from Branch-and-bound and cutting plane.

- Basically a branch-and-bound algorithm
- at each node solve LP-relaxation to find bound
- generate valid inequalities which separate the LP-solution, and which are valid for the whole problem
- maintain pool of valid inequalities
- branch when cuts become weak
- convergence ensured by branch-and-bound

Improvements

- Heuristic for generating cut
- Problem specific cuts
- Heuristic for removing cuts

If separation problem is “easy” the cut is not tight