Wednesday, November 10

Program of the day:

- Cutting planes — a method to obtain tighter bounds and faster convergence to integer solutions (Wolsey chap. 8)
- Application: branch-and-cut algorithms

Introduction

- branch-and-bound: divide and conquer.
- cutting plane: add inequalities which separate fractional solution from solution space.

Development

- 50's cutting plane (Gomory: simplex, no $\mathcal{NP}$-hardness)
- 70's tighten formulation in preprocessing
- 80-90's branch-and-cut (Padberg, Rinaldi)

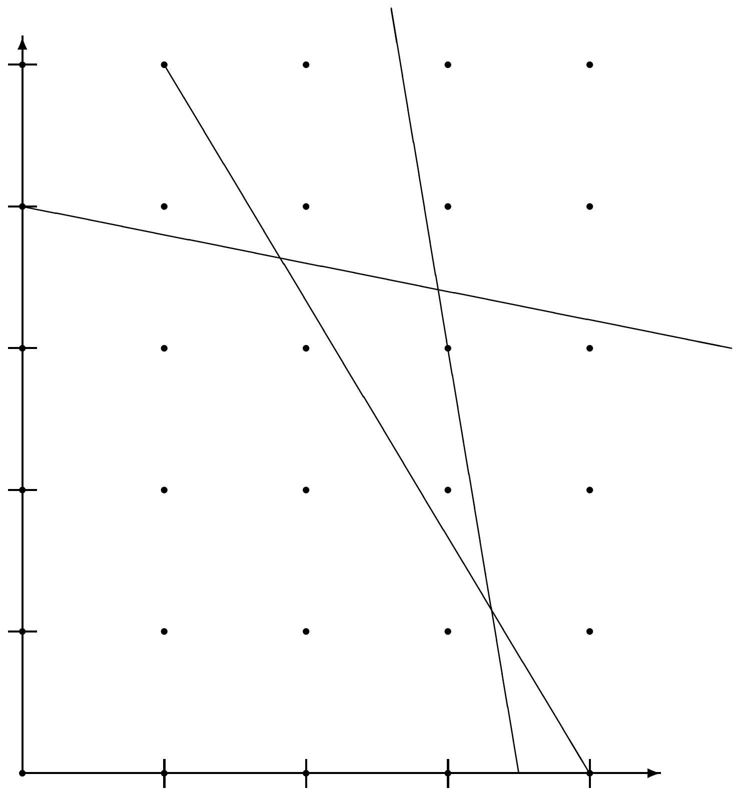
Preprocessing $\rightarrow$ part of solution process

Definitions

- cuts: valid inequalities
- facets: inequalities defining convex hull

Cuts and facets are redundant for IP formulation
Tighten formulation for LP relaxation

Cuts and facets



Examples from last lesson

Preprocessing, integer variables

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & \ldots \\
\text{subject to} & 7x_1 & + & 3x_2 & - & 4x_3 & - & 2x_4 & \leq & 1 \\
 & -2x_1 & + & 7x_2 & + & 3x_3 & + & 4x_4 & \leq & 6 \\
 & & - & 2x_2 & - & 3x_3 & - & 6x_4 & \leq & -5 \\
 & 3x_1 & & & - & 2x_3 & & & \geq & -1 \\
 & x & \in & \mathbb{B}^4
\end{array}
$$

Generating logical inequalities

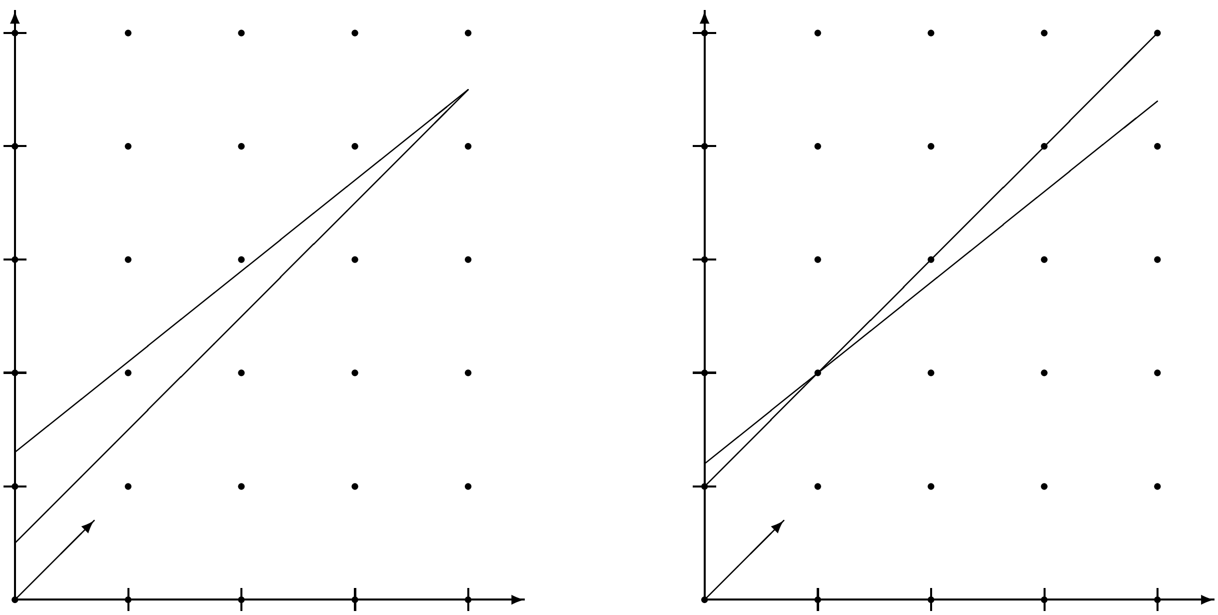
From constraint 1 we see that

- if $x_1 = 1$ and $x_2 = 1$ then infeasible, thus

$$x_1 + x_2 \leq 1$$

Examples from last lesson

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & -2x_1 + \;\; 2x_2 \geq 1 \\ & -8x_1 + 10x_2 \leq 13 \\ & x_1, x_2 \geq 0, \text{integer} \end{array}$$



Tightening formulation

$$\begin{array}{rcl} -2x_1 + 2x_2 & \geq & 1 \\ -x_1 + x_2 & \geq & 1/2 \\ -x_1 + x_2 & \geq & 1 \end{array} \qquad \begin{array}{rcl} -8x_1 + 10x_2 & \leq & 13 \\ -4x_1 + 5x_2 & \leq & 13/2 \\ -4x_1 + 5x_2 & \leq & 6 \end{array}$$

Motivation

Integer programming problem (IP)

$$\max\{cx : x \in X\}$$

where $X = \{x : Ax \leq b, x \in \mathbb{Z}_+^n\}$. Reformulate to

$$\max\{cx : x \in \text{conv}(X)\}$$

For any c, an optimal solution to LP is also optimal to IP

Valid inequalities (def. 8.1)

Consider the problem:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

An inequality

$$\pi x \leq \pi_0$$

is a *valid inequality* for $X \subseteq \mathbb{R}^n$ if

$$\pi x \leq \pi_0 \ \text{ for all } \ x \in X$$

Characterisation of valid inequalities (sec. 8.3.2)

Consider the problem:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

where

$$X = \{y \in \mathbb{Z} : y \leq b\}$$

then the inequality

$$y \leq \lfloor b \rfloor$$

is valid for X

- Simple observation
- Complete characterisation

Overview of cuts

- Chvatal cuts
- Gomory cuts (Modular cuts)
- Chvatal-Gomory cuts
- Disjunctive cuts
- Cover inequalities
- Clique inequalities
- Problem specific cuts

Notice

- Cuts and facets are independant of objective function
- A tight formulation can be used for any objective

Example of Facets

The problem

$$\begin{array}{lrcl} \text{minimize} & 2x_1+7x_2+2x_3 & & \\ \text{subject to} & x_1+4x_2+x_3 & \geq & 10 \\ & 4x_1+2x_2+2x_3 & \geq & 13 \\ & x_1+x_2-x_3 & \geq & 0 \\ & x_1,x_2,x_3 & \geq & 0,\ \text{integer} \end{array}$$

has the facets

$$\begin{array}{rcl} x_1+4x_2+x_3 & \geq & 10 \\ 2x_1+x_2+x_3 & \geq & 7 \\ x_1+x_2-x_3 & \geq & 0 \\ x_1+3x_2+x_3 & \geq & 9 \\ 2x_1+4x_2+x_3 & \geq & 13 \\ x_1+x_2+x_3 & \geq & 5 \\ x_1+2x_2 & \geq & 5 \\ 2x_1+x_2 & \geq & 4 \\ x_1 & \geq & 0,\ \text{integer} \\ x_2 & \geq & 0,\ \text{integer} \\ x_3 & \geq & 0,\ \text{integer} \end{array}$$

Using the new formulation we obtain an integer optimal solution by solving the LP-relaxed problem. (For any objective function).

Chvátal Cuts

Valid inequalities for a pure IP-model (minimization)

1 Add constraints, using suitable multipliers

2 Divide through by a common coefficient factor

3 Round up right-hand-side to the next integer

Example

$$
\begin{array}{llrcll}
\text{minimize} & & 2x_1 + 7x_2 + 2x_3 & & & \\
\text{subject to} & & x_1 + 4x_2 + x_3 & \geq & 10 & (1) \\
& & 4x_1 + 2x_2 + 2x_3 & \geq & 13 & (2) \\
& & x_1 + x_2 - x_3 & \geq & 0 & (3) \\
& & x_1 & \geq & 0 & (4) \\
& & x_2 & \geq & 0 & (5) \\
& & x_3 & \geq & 0 & (6) \\
& & x_1, x_2, x_3 \text{ integer} & & &
\end{array}
$$

1 times (2) is

$$4x_1 + 2x_2 + 2x_3 \geq 13$$

divide by two

$$2x_1 + x_2 + x_3 \geq 6\tfrac{1}{2}$$

left hand side is integral, thus round up right-hand

$$2x_1 + x_2 + x_3 \geq 7$$

Example (continued)

$$
\begin{array}{llcll}
\text{minimize} & 2x_1 + 7x_2 + 2x_3 & & & \\
\text{subject to} & x_1 + 4x_2 + x_3 & \geq & 10 & (1) \\
 & 4x_1 + 2x_2 + 2x_3 & \geq & 13 & (2) \\
 & x_1 + x_2 - x_3 & \geq & 0 & (3) \\
 & x_1 & \geq & 0 & (4) \\
 & x_2 & \geq & 0 & (5) \\
 & x_3 & \geq & 0 & (6) \\
 & x_1, x_2, x_3 \text{ integer} & & &
\end{array}
$$

Facets

$$
\begin{array}{rcll}
x_1 + 4x_2 + x_3 & \geq & 10 & \text{(a)} \\
2x_1 + x_2 + x_3 & \geq & 7 & \text{(b)} \\
x_1 + x_2 - x_3 & \geq & 0 & \text{(c)} \\
x_1 + 3x_2 + x_3 & \geq & 9 & \text{(d)} \\
2x_1 + 4x_2 + x_3 & \geq & 13 & \text{(e)} \\
x_1 + x_2 + x_3 & \geq & 5 & \text{(f)} \\
x_1 + 2x_2 & \geq & 5 & \text{(g)} \\
2x_1 + x_2 & \geq & 4 & \text{(h)} \\
x_1, x_2, x_3 & \geq & 0, \text{ integer} &
\end{array}
$$

Obtained as

(d) : 5 × (1), 1 × (b), 1 × (6), divide 7
(f) : 1 × (1), 3 × (b), 3 × (6), divide 7
(g) : 4 × (1), 1 × (b), 5 × (3), divide 11
(h) : 1 × (b), 1 × (c), 1 × (4), divide 2
(e) : 3 × (d), 1 × (b), 3 × (g), divide 4

Chvatal-Gomory cuts (p. 119)

$$\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{1j} x_j \leq b_1 \\
& \vdots \\
& \sum_{j=1}^{n} a_{mj} x_j \leq b_m \\
& x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{aligned}$$

1 Take a linear combination of the constraints

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} u_i a_{ij}\right) x_j \leq \left(\sum_{i=1}^{m} u_i b_i\right)$$

in short

$$\sum_{j=1}^{n} a'_j x_j \leq b'$$

2 Since $x \geq 0$ implies $\sum_{j=1}^{n} (a'_j - \lfloor a'_j \rfloor) x_j \geq 0$ we have

$$\sum_{j=1}^{n} \lfloor a'_j \rfloor x_j \leq b'$$

3 Since $x_j \in \mathbb{Z}_+$ implies $\lfloor a'_j \rfloor x_j \in \mathbb{Z}$ we get

$$\sum_{j=1}^{n} \lfloor a'_j \rfloor x_j \leq \lfloor b' \rfloor$$

Chvatal-Gomory (Theorem 8.4)

$$X = \{x : Ax \leq b, x \in \mathbb{Z}_+^n\}$$

Every valid inequality for X can be obtained by applying the Chvatal-Gomory procedure a finite number of times.

Notice

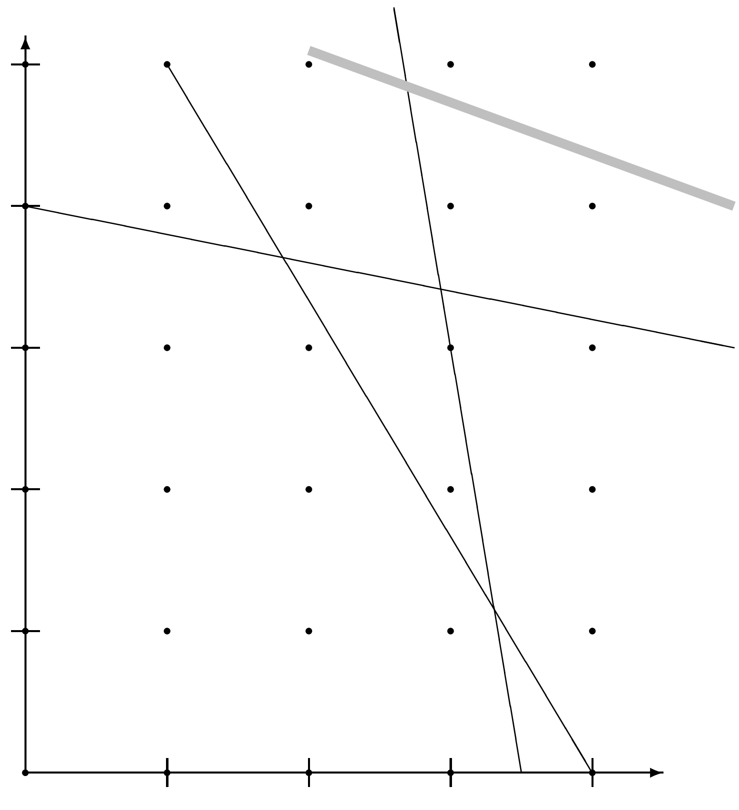
- No stronger inequalities than Chvatal-Gomory exists.
- Even the facet constraints can be generated as Chvatal-Gomory cuts.
- No constructive (polynomial) algorithm for how the linear combination of constraints should be chosen.
- In practice, the derivation of Chvatal-Gomory cuts must rely on specific features of a given application.

Gomory cuts is a systematical way of deriving cutting planes.

Only 0-1 case

All bounded integer variables can be expressed as sum of binary variables.

The set $X = P \cap \mathbb{Z}^n$



$P = \{x \in \mathbb{R}^n : Ax \leq b, 0 \leq x \leq 1\} \neq \emptyset$
$X = P \cap \mathbb{Z}^n$

Nemhauser and Wolsey, Proposition 1.1 page 208:

If inequality $\pi x \leq \pi_0$ is valid for P then it can be obtained as a C-G cut	(*)

- LP-redundant constraints are C-G inequalities
- Theorem 8.4 deals with IP-constraints

Proof (0-1 case)

Assume that

$$\pi x \leq \pi_0 \ \text{ where } \ \pi, \pi_0 \text{ integers}$$

is a valid inequality for X. We will show that this inequality can be obtained by using the C-G procedure a finite number of times.

- Step 1: Find a large number $t \in \mathbb{Z}_+$ such that

  $$\pi x \leq \pi_0 + t$$

  is a valid C-G inequality

- Step 2: Prove that if

  $$\pi x \leq \pi_0 + \tau + 1$$

  for $\tau \in \mathbb{Z}_+$ is a C-G inequality for X then also

  $$\pi x \leq \pi_0 + \tau$$

  is a C-G inequality for X.

- Step 3: Use step 2 for $\tau = t - 1, \ldots, 0$ each time getting a new C-G inequality

(Proof by induction)

Step 1

The inequality

$$\pi x \leq \pi_0 + t$$

is valid for P for some $t \in \mathbb{Z}_+$.

Proof

We have the inequality

$$x \leq 1$$

derive C-G inequality using multipliers $u = \pi$

$$\pi x \leq \pi 1$$

choosing $t = \pi 1 - \pi_0$ (π, π_0 is integer) we get the form

$$\pi x \leq \pi 1 = \pi_0 + t$$

for some $t \in \mathbb{Z}_+$

Note that $t < \infty$ as $P \subseteq [0,1]^n$ is bounded so $\max\{\pi x \mid x \in P\} < \infty$.

Step 2

Difficult part

a) Prove that if $\pi x \leq \pi_0 + \tau + 1$ with $\tau \in \mathbb{Z}_+$ is a C-G cut then

$$\pi x \leq \pi_0 + \tau + \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)$$

is a C-G inequality for X for every partition (N^0, N^1) of $N = \{1, \ldots, n\}$.

b) Use partitionings $(T^0 \cup \{n\}, T^1)$ and $(T^0, T^1 \cup \{n\})$ to obtain a new inequality for (T^0, T^1).

c) Derive all valid inequalities for partitionings of $N' = \{1, \ldots, n-1\}$

d) Repeating this procedure n times implies that we eliminate the sums on the right side and thus

$$\pi x \leq \pi_0 + \tau$$

is a C-G cut

Time complexity

- part (c) takes $O(2^n)$,
  part (d) is performed n times,
  in total $O(n2^n)$
- we run Step 2 $O(t)$ times, thus in total $O(tn2^n)$.

Step 2, a)

An inequality is valid for P if it is valid for all extreme points $\{x^1,\ldots,x^m\}$ of P

Assume that $\pi x \le \pi_0 + \tau + 1$ with $\tau \in \mathbb{Z}_+$ is a valid cut. Let (N^0, N^1) be any partitioning of $N = \{1,\ldots,n\}$. Consider an extreme point x^k of P

- x^k integer: then $\pi x^k \le \pi_0$ (since $\pi x \le \pi_0$ valid for X)
- x^k fractional: exists $\varepsilon > 0$ such that

$$\varepsilon^k \le \sum_{j\in N^0} x_j^k + \sum_{j\in N^1} (1 - x_j^k)$$

Choose $\alpha = \min_{x^k \text{ vertex in } P} \varepsilon^k$

Using $M \ge (\tau+1)/\alpha$, we have

$$\tau + 1 \le M\alpha \le M\left(\sum_{j\in N^0} x_j + \sum_{j\in N^1}(1-x_j)\right)$$

adding π_0 at both sides we get valid inequality for P

$$\pi x \le \pi_0 + \tau + 1 \le \pi_0 + M\left(\sum_{j\in N^0} x_j + \sum_{j\in N^1}(1-x_j)\right)$$

Due to (*) the inequality is a C-G cut.

Step 2, a)

We have just shown that the following is a C-G inequality

$$\pi x \leq \pi_0 + M \left(\sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j) \right)$$

By assumption we had the C-G inequality

$$\pi x \leq \pi_0 + \tau + 1$$

use weights $1/M$ and $(M-1)/M$ for the two inequalities getting C-G inequality

$$\pi x \leq \pi_0 + \tau + \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)$$

Step 2, b)

Use partitions $(T^0 \cup \{n\}, T^1)$ and $(T^0, T^1 \cup \{n\})$

$$\pi x \leq \pi_0 + \tau + \sum_{j \in T^0 \cup \{n\}} x_j + \sum_{j \in T^1} (1 - x_j)$$

and

$$\pi x \leq \pi_0 + \tau + \sum_{j \in T^0} x_j + \sum_{j \in T^1 \cup \{n\}} (1 - x_j)$$

using multipliers $1/2$ and $1/2$ we get C-G inequality

$$\pi x \leq \pi_0 + \tau + \sum_{j \in T^0} x_j + \sum_{j \in T^1} (1 - x_j)$$

Branch-and-cut algorithms

Combines best properties from Branch-and-bound and cutting plane.

- Basically a branch-and-bound algorithm
- at each node solve LP-relaxation to find bound
- generate valid inequalities which separate the LP-solution, and which are *valid for the whole problem*
- maintain pool of valid inequalities
- branch when cuts become weak
- convergence ensured by branch-and-bound

Improvements

- Heuristic for generating cut
- Problem specific cuts
- Heuristic for removing cuts

If separation problem is “easy” the cut is not tight