Friday, November 5

Program of the day: (Wolsey chapter 7)

- Branch-and-bound
- Hierarchy of techniques
- Preprocessing
- Example: A location problem solved through branch-and-bound

Remember: Dorit Hochbaum, tuesday 9/11 at 15.15

CPLEX

The strategy for solving MIP problems

- Preprocessing
- Additional constraints
- Best-first search in branch-and-bound
  (Space consuming)
- Dual simplex in every iteration
- Branch on variables close to integer solution
- Then branch on other variables ("the mess")
- CPLEX finds "approximate" solutions to MIP

Solving IP models — hierarchy of techniques

Some IP naturally lead to integer solutions

- Totally unimodular matrices
- Several transportations problems and network problems are totally unimodular.

Preprocessing and reformulation

- Reformulation of constraints to TU
- Tightening bounds
- Variable fixing
- Redundant constraints

Branch-and-bound methods

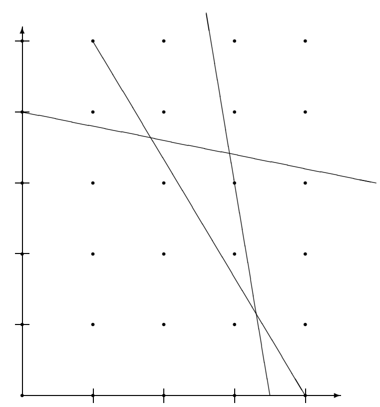
- Branching strategy
- Dual simplex

Preprocessing

LP: reduce number of constraints (LP-redundant), tighten bounds

IP: extend number of constraints (IP-redundant), closer to convex hull

Since the model will be solved several times during branch-and-bound, preprocessing will pay off.



Preprocessing

Example:

$$\begin{array}{lrcrcrcr} \text{maximize} & 2x_1 & + & x_2 & - & x_3 & & \\ \text{subject to} & 5x_1 & - & 2x_2 & + & 8x_3 & \leq & 15 \\ & 8x_1 & + & 3x_2 & - & x_3 & \geq & 9 \\ & x_1 & + & x_2 & + & x_3 & \leq & 6 \\ & 0 & \leq & x_1 & \leq & 3 & & \\ & 0 & \leq & x_2 & \leq & 1 & & \\ & 1 & \leq & x_3 & & & & \end{array}$$

Tightening bounds

Isolating x_1 in the first constraint

$$5x_1 \leq 15 + 2x_2 - 8x_3 \leq 15 + 2 - 8 = 9$$

thus $x_1 \leq \frac{9}{5}$.
Isolating x_3 in the first constraint

$$8x_3 \leq 15 + 2x_2 - 5x_1 \leq 15 + 2 - 0 = 17$$

thus $x_3 \leq \frac{17}{8}$.

In this way, pass through all variables, all constraints. Time complexity $O(mn^2)$. May be repeated several times.

Preprocessing, tightening bounds

$$\begin{array}{ll} \text{maximize} & \ldots \\ \text{subject to} & a_0 x_0 + \sum_{j=1}^n a_j x_j \leq b \\ & \ell_j \leq x_j \leq u_j \end{array}$$

Then we have the additional bounds (which may be tighter)

- If $a_0 > 0$ then

$$x_0 \leq \frac{1}{a_0}\left(b - \sum_{j:a_j>0} a_j \ell_j - \sum_{j:a_j<0} a_j u_j\right)$$

- If $a_0 < 0$ then

$$x_0 \geq \frac{1}{a_0}\left(b - \sum_{j:a_j>0} a_j \ell_j - \sum_{j:a_j<0} a_j u_j\right)$$

Preprocessing, redundant constraints

$$\begin{array}{lrcrcrcr} \text{maximize} & 2x_1 & + & x_2 & - & x_3 & & \\ \text{subject to} & 5x_1 & - & 2x_2 & + & 8x_3 & \leq & 15 \\ & 8x_1 & + & 3x_2 & - & x_3 & \geq & 9 \\ & x_1 & + & x_2 & + & x_3 & \leq & 6 \\ & \frac{7}{8} & \leq & x_1 & \leq & \frac{9}{5} & & \\ & 0 & \leq & x_2 & \leq & 1 & & \\ & 1 & \leq & x_3 & \leq & \frac{101}{64} & & \end{array}$$

Considering constraint 3 we get

$$x_1 + x_2 + x_3 \leq \frac{9}{5} + 1 + \frac{101}{64} < 6$$

thus the constraint is redundant

Preprocessing, redundant constraints

$$\begin{array}{ll} \text{maximize} & \ldots \\ \text{subject to} & \sum_{j=0}^n a_j x_j \leq b \\ & \ell_j \leq x_j \leq u_j \end{array}$$

The constraint $\sum_{j=0}^n a_j x_j \leq b$ is *redundant* if

$$\sum_{j:a_j>0} a_j u_j + \sum_{j:a_j<0} a_j \ell_j \leq b$$

The problem is *infeasible* if

$$\sum_{j:a_j>0} a_j \ell_j + \sum_{j:a_j<0} a_j u_j > b$$

Preprocessing, integer variables

$$\begin{array}{lrrrrl}
\text{maximize} & \ldots & & & & \\
\text{subject to} & 7x_1 & + 3x_2 & - 4x_3 & - 2x_4 & \leq 1 \\
& -2x_1 & + 7x_2 & + 3x_3 & + 4x_4 & \leq 6 \\
& & - 2x_2 & - 3x_3 & - 6x_4 & \leq -5 \\
& 3x_1 & & - 2x_3 & & \geq -1 \\
& x \in \mathbb{B}^4 & & & &
\end{array}$$

Generating logical inequalities

From constraint 1 we see that

- if $x_1 = 1$ then $x_3 = 1$, thus $x_1 \leq x_3$
- if $x_1 = 1$ then $x_4 = 1$, thus $x_1 \leq x_4$
- if $x_1 = 1$ and $x_2 = 1$ then infeasible, thus $x_1 + x_2 \leq 1$

From constraint 2 we see that

- if $x_2 = 1$ then $x_1 = 1$, thus $x_2 \leq x_1$

Combining inequalities

We have $x_1 + x_2 \leq 1$ and $x_2 \leq x_1$ thus $x_2 = 0$.

Simplifying

Insert $x_2 = 0$ in the model and repeat.

Preprocessing, integer variables

Variable fixation (maximization problem)

- For each variable x
- Consider the possible values of x as if we ran branch-and-bound algorithm starting with variable x
- If a branch has $\bar{z} \leq z$, drop this branch

Preprocessing, integer variables

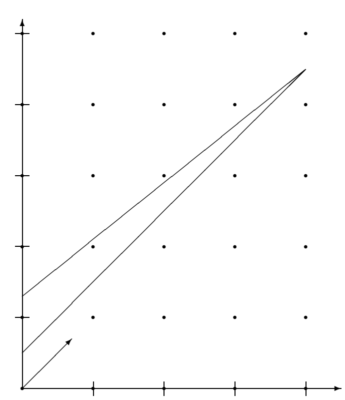
$$\begin{array}{lrrrrl}
\text{maximize} & x_1 & + x_2 & + x_3 & + x_4 & \\
\text{subject to} & 7x_1 & + 3x_2 & - 4x_3 & - 2x_4 & \leq 1 \\
& -2x_1 & + 7x_2 & + 3x_3 & + 4x_4 & \leq 6 \\
& & - 2x_2 & - 3x_3 & - 6x_4 & \leq -5 \\
& 3x_1 & & - 2x_3 & & \geq -1 \\
& x \in \mathbb{B}^4 & & & &
\end{array}$$

Assume that a solution with $z = 2$ has been guessed

- root, upper bound 3.29091
- $x_1 = 0$, upper bound 2
- $x_1 = 1$, upper bound 3
- $x_2 = 0$, upper bound 3
- $x_2 = 1$, infeasible
- $x_3 = 0$, upper bound 1.85454
- $x_3 = 1$, upper bound 3.29091
- $x_4 = 0$, infeasible
- $x_4 = 1$, upper bound 3.29091

Preprocessing

$$\begin{array}{ll}
\text{maximize} & x_1 + x_2 \\
\text{subject to} & -2x_1 + 2x_2 \geq 1 \\
& -8x_1 + 10x_2 \leq 13 \\
& x_1, x_2 \geq 0, \text{integer}
\end{array}$$



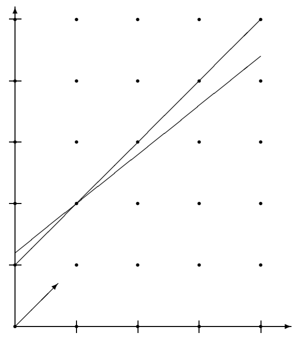
Preprocessing

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & -2x_1 + \ \ 2x_2 \geq 1 \\ & -8x_1 + 10x_2 \leq 13 \\ & x_1, x_2 \geq 0, \text{integer} \end{array}$$

preprocess:

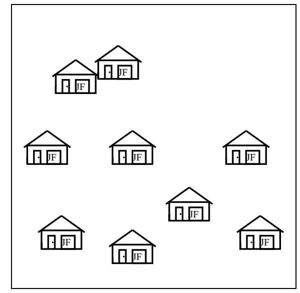
$$\begin{array}{rcl} -2x_1 + 2x_2 & \geq & 1 \\ -x_1 + x_2 & \geq & 1/2 \\ -x_1 + x_2 & \geq & 1 \end{array}$$

$$\begin{array}{rcl} -8x_1 + 10x_2 & \leq & 13 \\ -4x_1 + 5x_2 & \leq & 13/2 \\ -4x_1 + 5x_2 & \leq & 6 \end{array}$$



Example: A location problem

Dispersion problem: Open p out of n possible facilities so that their overall distance is maximized



- Distance i to j is $d_{ij} \geq 0$.
- $d_{ij} = d_{ji}$ and $d_{jj} = 0$.
- Binary variable x_j is one if facility open

p-dispersion problem

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} \sum_{i=1}^{n} d_{ij} x_i x_j \\ \text{subject to} & \sum_{j=1}^{n} x_j = p \\ & x_j \in \{0,1\}, \quad j = 1, \ldots, n. \end{array}$$

p dispersion problem, example

Example:

$i \backslash j$	1	2	3	4	5	6	7
1	0	3	7	4	10	5	7
2	3	0	9	5	5	10	6
3	7	9	0	1	3	2	4
4	4	5	1	0	1	9	1
5	10	5	3	1	0	3	2
6	5	10	2	9	3	0	3
7	7	6	4	1	2	3	0

$n = 7$, $p = 3$.

Optimal solution is $x_2 = x_4 = x_6 = 1$, objective 48.

Linear formulation

Quadratic model

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} \sum_{i=1}^{n} d_{ij} x_i x_j \\ \text{subject to} & \sum_{j=1}^{n} x_j = p \\ & x_j \in \{0,1\} \end{array}$$

Introduce $y_{ij} = 1 \Leftrightarrow (x_i = 1 \text{ and } x_j = 1)$.

$$y_{ij} \leq x_i, \qquad y_{ij} \leq x_j, \qquad x_i + x_j \leq 1 + y_{ij}$$

Linear model

$$\begin{array}{lll} \text{maximize} & \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} y_{ij} & \\ \text{subject to} & \sum_{j=1}^{n} x_j = p & \\ & y_{ij} \leq x_i & j = 1, \ldots, n \\ & y_{ij} \leq x_j & i = 1, \ldots, n \\ & x_i + x_j \leq 1 + y_{ij} & i, j = 1, \ldots, n \\ & x_j, y_{ij} \in \{0,1\} & \end{array}$$

Constraint $y_{ij} = 1 \Leftarrow (x_i = 1 \text{ and } x_j = 1)$ not necessary.

Better linear formulation

Quadratic model

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij}x_ix_j \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = p \\ & x_j \in \{0,1\} \end{aligned}$$

Constraint $y_{ij} = 1 \Leftarrow (x_i = 1 \text{ and } x_j = 1)$ not necessary.
Introduce $(y_{ij} = 1 \Rightarrow x_j = 1)$ and $(y_{ij} = 1 \Leftrightarrow y_{ji} = 1)$

$$y_{ij} \le x_j, \qquad y_{ij} = y_{ji},$$

Multiply $\sum_{i=1}^{n} x_i = p$ by x_j for each j getting

$$\sum_{i=1}^{n} x_ix_j = \sum_{i=1}^{n} y_{ij} = px_j \qquad j = 1,\ldots,n$$

Linear model

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}y_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = p \\ & \sum_{i=1}^{n} y_{ij} = px_j & j = 1,\ldots,n \\ & y_{ij} = y_{ji} & i,j = 1,\ldots,n \\ & y_{ij} \le x_j & i,j = 1,\ldots,n \\ & x_j, y_{ij} \in \{0,1\} \end{aligned}$$

Relaxation: drop $y_{ij} = y_{ji}$

Better linear formulation

Relaxed linear model

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}y_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = p \\ & \sum_{i=1}^{n} y_{ij} = px_j & j = 1,\ldots,n \\ & y_{ij} \le x_j & i,j = 1,\ldots,n \\ & x_j, y_{ij} \in \{0,1\} \end{aligned}$$

$$\begin{array}{lllllllll} \max & \boxed{\sum_{i=1}^{n} d_{i1}y_{i1}} + & \boxed{\sum_{i=1}^{n} d_{i2}y_{i2}} + & \boxed{\sum_{i=1}^{n} d_{i3}y_{i3}} & + \ldots + 0x_1 & + 0x_2 & + 0x_3 & + \ldots & \\ \text{s.t.} & \boxed{\sum_{i=1}^{n} y_{i1}} & & & -px_1 & & & & = 0 \\ & & \boxed{\sum_{i=1}^{n} y_{i2}} & & & -px_2 & & & = 0 \\ & & & \boxed{\sum_{i=1}^{n} y_{i3}} & & & -px_3 & & = 0 \\ & & & & x_1 & +x_2 & +x_3 & + \ldots & = p \end{array}$$

p dispersion problem, deriving the bound

$i \backslash j$	1	2	3	4	5	6	7
1	0	3	7	4	10	5	7
2	3	0	9	5	5	10	6
3	7	9	0	1	3	2	4
4	4	5	1	0	1	9	1
5	10	5	3	1	0	3	2
6	5	10	2	9	3	0	3
7	7	6	4	1	2	3	0

$n = 7$, $p = 3$.

$d'_1 = 24$ $d'_2 = 25$ $d'_3 = 20$ $d'_4 = 18$ $d'_5 = 18$ $d'_6 = 24$ $d'_7 = 17$

Upper bound d'_j on each facility j

$$\begin{aligned} \text{maximize} \quad & d'_j = \sum_{i=1}^{n} d_{ij}y_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{n} y_{ij} = p \\ & y_{ij} \in \{0,1\}, \quad i = 1,\ldots,n. \end{aligned}$$

Upper bound $\bar{z}$

$$\begin{aligned} \text{maximize} \quad & \bar{z} = \sum_{j=1}^{n} d'_jx_j \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = p \\ & x_j \in \{0,1\}, \quad j = 1,\ldots,n. \end{aligned}$$

Branch-and-bound algorithm

- Depth first search (bounds in $O(n)$ time)
- Branch by splitting on x_j
- Fixed order of variables according to d'_j
- Branch $x_j = 0$: remove row, column j
- Branch $x_j = 1$: decrease p, add d_{jj} to objective, set $d_{ii} := d_{ii} + d_{ij} + d_{ji}$ for $i = j+1, \dots, n$. remove row, column j.

Branch-and-bound tests

GEO *geometrical problems*
d_{ij} euclidean distance between i and j

WGEO *weighted geometrical problems*
Each facility has a weight. d_{ij} euclidean distance between i and j times weights

EXP *exponential distribution*
d_{ij} with $i < j$ is randomly drawn from exponential distribution.

AEXP *asymmetric exponential distribution*
as above but $d_{ij} \neq d_{ji}$

RAN *random distances*
d_{ij} randomly distributed in $[1 \dots 100]$.

DSUB *dense subgraph*
d_{ij} is set to 1 or 0 with 50% probability.

Branch-and-bound results

n	GEO	WGEO	EXP	AEXP	RAN	DSUB
10	7.66	6.76	7.90	7.57	7.68	8.30
20	9.04	4.03	11.11	11.60	11.94	16.37
30	9.15	3.25	14.84	13.40	12.88	16.58
40	8.59	1.47	14.97	13.45	8.64	11.24
50	9.07	3.39	16.17	9.77	20.89	27.86
60	19.69	2.68	22.31	15.69	15.52	—
70	10.80	3.20	—	—	—	—
80	6.93	2.76	—	—	—	—
90	—	2.92	—	—	—	—
100	—	7.09	—	—	—	—
150	—	2.38	—	—	—	—
200	—	2.78	—	—	—	—

Table 1: Relative deviation of upper bound $\bar{z}$ in pct. Average of 10 instances.

n	GEO	WGEO	EXP	AEXP	RAN	DSUB
10	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00
30	0.01	0.01	0.01	0.01	0.02	0.02
40	0.05	0.01	0.11	0.31	0.57	1.60
50	0.64	0.03	2.79	0.58	12.65	30.47
60	2.85	0.05	87.28	61.52	4552.81	—
70	39.18	0.09	—	—	—	—
80	153.15	0.17	—	—	—	—
90	—	0.33	—	—	—	—
100	—	0.44	—	—	—	—
150	—	3.08	—	—	—	—
200	—	161.21	—	—	—	—

Table 2: Solution times in seconds as average of 10 instances.

Note:

- Bounds with gap more than 1 – 2% are of little use
- Quadratic problems are difficult to solve through ILP

n	GEO	WGEO	EXP	AEXP	RAN	DSUB
10	12	7	8	7	9	6
20	140	18	81	227	328	448
30	1654	45	2082	2659	5716	8420
40	12675	26	42851	141162	276980	927312
50	220355	858	1105817	218292	5565562	16162737
60	816524	554	28536918	19629045	3217643	—
70	9727736	1241	—	—	—	—
80	28711239	7282	—	—	—	—
90	—	16652	—	—	—	—
100	—	13646	—	—	—	—
150	—	123478	—	—	—	—
200	—	7302184	—	—	—	—

Table 3: Number of branch-and-bound nodes. Average of 10 instances.