Multiple-choice knapsack problem

We are given a number of classes $N_1, \dots, N_k$ of items. Each item $j \in N_i$ has an associated profit p_{ij} and a weight w_{ij}. The objective of the problem is to choose exactly one item from each class N_i such that the profit sum of the chosen items is maximized, while the weight sum of the chosen items cannot exceed a given capacity c.

In the following instance we have $k = 3$ classes, and the capacity is $c = 9$.

$N_1 = \{1,2,3\}$ $\quad N_2 = \{1,2\}$ $\quad N_3 = \{1,2,3\}$

j	1	2	3
$p_{1,j}$	0	4	6
$w_{1,j}$	0	3	4

j	1	2
$p_{2,j}$	2	3
$w_{2,j}$	1	2

j	1	2	3
$p_{3,j}$	0	3	4
$w_{3,j}$	3	4	8

Q11: Solve the above problem to integer optimality. What is the optimal solution value z?

11A) $z = 8$

11B) $z = 9$

11C) $z = 10$

11D) $z = 11$

11E) $z = 12$

11F) $z = 13$

Q12: We introduce the binary variables x_{ij} to indicate if item j is chosen in class N_i. What is the correct formulation of the multiple-choice knapsack problem?

12A)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j\in N_i} p_{ij}x_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j\in N_i} w_{ij}x_{ij} \le c \\ & \textstyle\sum_{j\in N_i} x_{ij} = 1, && i = 1,\dots,k \\ & x_{ij} \in \{0,1\}, && i = 1,\dots,k,\ j \in N_i \end{aligned}$$

12B)
$$\begin{aligned} \max\ & \textstyle\sum_{j\in N_i} p_{ij}x_{ij}, && i = 1,\dots,k \\ \text{s.t.}\ & \textstyle\sum_{j\in N_i} w_{ij}x_{ij} \le c, && i = 1,\dots,k \\ & x_{ij} = 1, && i = 1,\dots,k,\ j \in N_i \\ & x_{ij} \in \{0,1\}, && i = 1,\dots,k,\ j \in N_i \end{aligned}$$

12C)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j\in N_i} p_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j\in N_i} w_{ij} \le c \\ & \textstyle\sum_{j\in N_i} x_{ij} = 1, && i = 1,\dots,k \\ & x_{ij} \in \{0,1\}, && i = 1,\dots,k,\ j \in N_i \end{aligned}$$

12D)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j\in N_i} p_{ij}x_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j\in N_i} w_{ij}x_{ij} \le c \\ & \textstyle\sum_{i=1}^{k} x_{ij} = 1, && j \in N_i \\ & x_{ij} \in \{0,1\}, && i = 1,\dots,k,\ j \in N_i \end{aligned}$$

12E)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j=1}^{k} p_{ij}x_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j=1}^{k} w_{ij}x_{ij} \le c \\ & \textstyle\sum_{i=1}^{k} x_{ij} = 1, && j \in N_i \\ & x_{ij} \in \{0,1\}, && i = 1,\dots,k,\ j \in N_i \end{aligned}$$

12F)
$$\begin{aligned} \max\ & \textstyle\sum_{i=1}^{k} \sum_{j\in N_i} p_{ij}x_{ij} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{k} \sum_{j\in N_i} w_{ij}x_{ij} \le c \\ & \textstyle\sum_{i=1}^{k} \sum_{j\in N_i} x_{ij} = 1 \\ & x_{ij} \in \{0,1\}, && i = 1,\dots,k,\ j \in N_i \end{aligned}$$

In order to tighten the formulation we would like to derive a cover. A cover $C = \{h_1, \dots, h_k\}$ consists of one index h_i from each class N_i such that

$$\sum_{i=1}^{k} w_{ih_i} > c$$

If C is a cover, we may impose a cover inequality of the form

$$\sum_{i=1}^{k} x_{ih_i} \le d$$

Q13: What is the smallest value of d such that the above cover inequality is valid?

13A) $d = c$
13B) $d = |N_i|$
13C) $d = k$
13D) $d = c - 1$
13E) $d = |N_i| - 1$
13F) $d = k - 1$

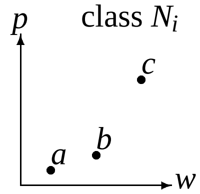
Q14: **(text question)** If the LP-solution (x^*_{ij}) to the multiple-choice knapsack problem is non-integral, describe a separation algorithm which separates the most violated cover inequality. (*Hint:* See Wolsey Section 9.3.3 for the “normal” cover inequalities).

Q15: **(text question)** Prove that if we have three items a, b and c in the same class N_i such that $w_{ia} \le w_{ib} \le w_{ic}$ and

$$\frac{p_{ic} - p_{ia}}{w_{ic} - w_{ia}} \ge \frac{p_{ib} - p_{ia}}{w_{ib} - w_{ia}}$$

then an optimal LP-solution exists where $x_{ib} = 0$. (See the following example for a geometrical interpretation).

class N_i



j	a	b	c
p_{ij}	1	2	7
w_{ij}	2	5	8

Integer programming

Consider the following problem

$$\begin{array}{lrcrcrl}
\text{maximize} & x_1 & + & 2x_2 & & & \\
\text{subject to} & 4x_1 & + & x_2 & \le & 8 & \text{(a)} \\
 & x_1 & - & x_2 & \ge & -1 & \text{(b)} \\
 & x_1 & & & \le & 2 & \text{(c)} \\
 & & & x_2 & \le & 2 & \text{(d)} \\
 & x_1 & & & \ge & 0 & \text{(e)} \\
 & & & x_2 & \ge & 0 & \text{(f)} \\
 & x_1, x_2 \in \mathbb{Z} & & & & &
\end{array}$$

Q16: What is the largest set of facet-defining inequalities?

16A) all of the constraints (a) to (f)
16B) (d), (e), (f)
16C) (b), (d), (e), (f)
16D) (c), (d), (e), (f)
16E) (b)
16F) (b), (e), (f)

Q17: Assume that we solve the problem to LP-optimality. What is the value of the dual variables y_a, y_d corresponding to constraints (a) and (d).

17A) $y_a = \frac{1}{2}, y_d = \frac{3}{7}$

17B) $y_a = 2, y_d = 0$

17C) $y_a = \frac{1}{4}, y_d = \frac{2}{3}$

17D) $y_a = \frac{1}{4}, y_d = \frac{7}{4}$

17E) $y_a = \frac{1}{3}, y_d = \frac{4}{3}$

17F) $y_a = \frac{1}{4}, y_d = \frac{1}{3}$

Q18: Assume that we Lagrangian relax constraints (a) and (b) using multipliers λ_a and λ_b respectively. What is the optimal value of the Lagrangian multipliers when solving the Lagrangian dual?

18A) $\lambda_a = \frac{1}{2}, \lambda_b = 2$

18B) $\lambda_a = \frac{1}{2}, \lambda_b = 0$

18C) $\lambda_a = \frac{1}{3}, \lambda_b = \frac{1}{4}$

18D) $\lambda_a = \frac{1}{2}, \lambda_b = 3$

18E) $\lambda_a = \frac{1}{4}, \lambda_b = 0$

18F) $\lambda_a = \frac{1}{4}, \lambda_b = 3$

Model Building

An air cargo company is planning the transportation of some goods from city A to city B. Every plane can carry at most q weight units. A number of different items $j = 1, \ldots, n$, need to be transported, item j taking up w_j weight units. Each plane $i = 1, \ldots, n$ should contain at least 4 different items to ensure proper balancing. The air cargo company wishes to minimize the number of planes used.

Q19: The problem is formulated as an integer-programming model, in which $y_i = 1$ if and only if plane i is used, and $x_{ij} = 1$ if and only if item j is sent by plane i. Which of the following models is a correct formulation of the problem

19A)
$$\begin{aligned} \min\ & \textstyle\sum_{i=1}^n y_i \\ \text{s.t.}\ & \textstyle\sum_{j=1}^n w_j x_{ij} \le y_i q, && i = 1, \ldots, n \\ & \textstyle\sum_{j=1}^n x_{ij} \ge 4, && i = 1, \ldots, n \\ & \textstyle\sum_{i=1}^n x_{ij} = 1, && j = 1, \ldots, n \\ & x_{ij}, y_i \in \{0,1\}, && i, j = 1, \ldots, n \end{aligned}$$

19B)
$$\begin{aligned} \min\ & \textstyle\sum_{i=1}^n y_i \\ \text{s.t.}\ & \textstyle\sum_{j=1}^n w_j x_{ij} \le y_i q, && i = 1, \ldots, n \\ & \textstyle\sum_{j=1}^n x_{ij} \ge 4, && i = 1, \ldots, n \\ & \textstyle\sum_{j=1}^n x_{ij} = 1, && i = 1, \ldots, n \\ & x_{ij}, y_i \in \{0,1\}, && i, j = 1, \ldots, n \end{aligned}$$

19C)
$$\begin{aligned} \min\ & \textstyle\sum_{i=1}^n y_i \\ \text{s.t.}\ & \textstyle\sum_{j=1}^n w_j x_{ij} \le y_i q, && i = 1, \ldots, n \\ & \textstyle\sum_{i=1}^n x_{ij} = 1, && j = 1, \ldots, n \\ & x_{ij}, y_i \in \{0,1\}, && i, j = 1, \ldots, n \end{aligned}$$

19D)
$$\begin{aligned} \min\ & \textstyle\sum_{i=1}^n y_i \\ \text{s.t.}\ & \textstyle\sum_{i=1}^n w_j x_{ij} \le q \\ & \textstyle\sum_{i=1}^n x_{ij} \ge 4, && i = 1, \ldots, n \\ & \textstyle\sum_{i=1}^n x_{ij} = 1, && j = 1, \ldots, n \\ & x_{ij}, y_i \in \{0,1\}, && i, j = 1, \ldots, n \end{aligned}$$

19E)
$$\begin{aligned} \min\ & \textstyle\sum_{i=1}^n y_i \\ \text{s.t.}\ & \textstyle\sum_{i=1}^n w_j x_{ij} \le q \\ & \textstyle\sum_{i=1}^n x_{ij} \ge 4, && i = 1, \ldots, n \\ & \textstyle\sum_{j=1}^n x_{ij} = 1, && i = 1, \ldots, n \\ & x_{ij}, y_i \in \{0,1\}, && i, j = 1, \ldots, n \end{aligned}$$

19F)
$$\begin{aligned} \min\ & y_i \\ \text{s.t.}\ & \textstyle\sum_{i=1}^n w_j x_{ij} \le q \\ & \textstyle\sum_{i=1}^n x_{ij} \ge 4, && i = 1, \ldots, n \\ & \textstyle\sum_{i=1}^n x_{ij} = 1, && j = 1, \ldots, n \\ & x_{ij}, y_i \in \{0,1\}, && i, j = 1, \ldots, n \end{aligned}$$

Q20: For security reasons, it is decided that either items 1 and 2 should go by the same plane, or items 1, 2, 3 should go by different planes. What is the correct integer linear formulation of the constraints?

20A)
$$\begin{array}{ll} x_{i1} - x_{i2} = 0 & i = 1, \ldots, n \\ x_{i1} + x_{i2} \leq 1 & i = 1, \ldots, n \\ x_{i1} + x_{i3} \leq 1 & i = 1, \ldots, n \\ x_{i2} + x_{i3} \leq 1 & i = 1, \ldots, n \\ \delta \in \{0, 1\} & \end{array}$$

20B)
$$\begin{array}{ll} \delta(x_{i1} - x_{i2}) = 0 & i = 1, \ldots, n \\ (1-\delta)(x_{i1} + x_{i2} \leq 1) & i = 1, \ldots, n \\ (1-\delta)(x_{i1} + x_{i3} \leq 1) & i = 1, \ldots, n \\ (1-\delta)(x_{i2} + x_{i3} \leq 1) & i = 1, \ldots, n \\ \delta \in \{0, 1\} & \end{array}$$

20C)
$$\begin{array}{ll} x_{i1} - x_{i2} \leq \delta & i = 1, \ldots, n \\ x_{i1} - x_{i2} \geq \delta & i = 1, \ldots, n \\ x_{i1} + x_{i2} + \leq 2 - \delta & i = 1, \ldots, n \\ x_{i1} + x_{i3} + \leq 2 - \delta & i = 1, \ldots, n \\ x_{i2} + x_{i3} + \leq 2 - \delta & i = 1, \ldots, n \\ \delta \in \{0, 1\} & \end{array}$$

20D)
$$\begin{array}{ll} x_{i1} - x_{i2} + \delta \leq 1 & i = 1, \ldots, n \\ x_{i1} - x_{i2} - \delta \geq -1 & i = 1, \ldots, n \\ x_{i1} + x_{i2} + (1-\delta) \leq 2 & i = 1, \ldots, n \\ x_{i1} + x_{i3} + (1-\delta) \leq 2 & i = 1, \ldots, n \\ x_{i2} + x_{i3} + (1-\delta) \leq 2 & i = 1, \ldots, n \\ \delta \in \{0, 1\} & \end{array}$$

20E)
$$\begin{array}{ll} x_{i1} - x_{i2} + \delta \leq 1 & i = 1, \ldots, n \\ x_{i1} - x_{i2} - \delta \geq -1 & i = 1, \ldots, n \\ x_{i1} + x_{i2} + \delta \leq 2 & i = 1, \ldots, n \\ x_{i1} + x_{i3} + \delta \leq 2 & i = 1, \ldots, n \\ x_{i2} + x_{i3} + \delta \leq 2 & i = 1, \ldots, n \\ \delta \in \{0, 1\} & \end{array}$$

20F)
$$\begin{array}{ll} x_{i1} - x_{i2} + \delta_1 \leq 1 & i = 1, \ldots, n \\ x_{i1} - x_{i2} - \delta_1 \leq -1 & i = 1, \ldots, n \\ x_{i1} + x_{i2} + \delta_2 \leq 2 & i = 1, \ldots, n \\ x_{i1} + x_{i3} + \delta_2 \leq 2 & i = 1, \ldots, n \\ x_{i2} + x_{i3} + \delta_2 \leq 2 & i = 1, \ldots, n \\ \delta_1, \delta_2 \in \{0, 1\} & \end{array}$$

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