Introduction to Optimization:

Written Exam, 17 December 1998

Your assignment

20 different questions Q1-Q20 are posed on the subsequent pages which also provide you with the supplementary information when needed.

For each question a number of possible answers is given. Tick the appropriate box and note that at most one box must be marked for each question since your answer to the question will otherwise be disregarded

Note: only the last 10 questions are available

Questions Q11-Q13: Planning pension

Three professors, Dantzig, Gomory and Chvátal are sitting at the same table during a conference dinner. They enjoy the food very much, and in particular the excellent wine. "I wish that we could afford a good Burgondy every day for the rest of our life", says Chvátal. Professor Dantzig agrees: "Well, if we invest our pension properly, we could actually do this. As a matter of fact, I am using OR models to maximize the profit of my invested pension".

Professor Dantzig can invest his pension in three different projects. The amount of money invested in project i is denoted x_i. In order to avoid paying taxes of his profit, the investment must satisfy the following constraints

$$\begin{array}{lrrrcl} \text{max} & x_1 & +x_2 & +x_3 & & \\ \text{s.t.} & 2x_1 & +4x_2 & +7x_3 & \leq & 6 \\ & x_1 & -3x_2 & +x_3 & \leq & 8 \\ & x_1 & +x_2 & -x_3 & \leq & 1 \\ & \multicolumn{3}{l}{x_1, x_2, x_3 \geq 0} & & \end{array}$$

With a little bit of writing on his serviette, Professor Dantzig solves the problem by using the simplex algorithm. But in order to test the other professors, he only announces the dual solution $y_1 = \frac{2}{9}, y_2 = 0, y_3 = \frac{5}{9}$.

Q 11 What is the optimal primal solution of the above LP-problem (variables not listed are assumed to be zero)

11.a) $x_1 = \frac{13}{9}$, $x_2 = \frac{5}{9}$.

11.b) $x_2 = \frac{17}{9}$.

11.c) $x_1 = x_2 = x_3 = 1$.

11.d) $x_1 = \frac{13}{9}$, $x_3 = \frac{4}{9}$.

11.e) $x_3 = \frac{13}{9}$.

□

Professor Chvátal has been drinking too much of the good Burgondy and his nose is starting to shine in a pink red color. Looking at the solution he says: "This is not good enough. All our variables must attain integer values".

The other professors agree, but unfortunately this makes the problem considerably more difficult to solve. Professor Gomory however proposes an inequality which might make the problem easier to solve.

Q 12 One of the lines in the simplex tableau looks as:

$$x_1 = \frac{13}{9} - \frac{11}{9}x_2 - \frac{1}{9}s_1 - \frac{7}{9}s_3$$

where s_i is the slack variable of constraint i. Derive a Gomory cut from the above equation, using the formula from Nemhauser and Wolsey. Which one of the following inequalities appears?

12.a) $18x_1 + 27x_2 + 54x3 \leq 43$

12.b) $x_1 + x_2 \leq 1$

12.c) $\frac{1}{9}x_1 + \frac{8}{9}x_3 \geq 3$

12.d) $\frac{1}{3}x_2 + \frac{3}{4}x_3 \leq \frac{1}{7}$

12.e) $x_1 + 2x_2 + x_3 \leq 5$

12.f) $\frac{4}{9} + \frac{7}{9}x_1 + \frac{8}{9}x_2 + \frac{2}{9}x_3 \geq 1$

□

In the meantime, professor Chvátal has found the valid inequality

$$2x_1 + 2x_2 + 3x_3 \leq 5$$

Q 13 How can this inequality be obtained.

13.a) The inequality is a Chvátal-cut obtained by using multipliers $(2, 2, 1)$.

13.b) The inequality is a Chvátal-cut obtained by using multipliers $(\frac{1}{2}, \frac{3}{2}, \frac{1}{2})$.

13.c) The inequality is a Chvátal-cut obtained by using multipliers $(0, \frac{1}{2}, \frac{2}{3})$.

13.d) The inequality is a Chvátal-cut obtained by using multipliers $(2, 1, 3)$.

13.e) The inequality is a Chvátal-cut obtained by using multipliers $(1, 1, 2)$.

13.f) The inequality is not valid since it excludes integer feasible solutions.

□

Questions Q14-Q16: Still active

As professors get older, they get more concerned about which projects they should become engaged in. Professor P7 is planning his next semester, where he would like to work on five different projects $1, 2, 3, 4, 5$. Every week, he has at most 37 hours available for working on the projects. If he chooses to work on any project i, then he must work at least 5 hours on that project every week. Project 2 is in cooperation with professor X and project 3 involves professor Y. Since the two professors X and Y cannot stand each other, it will not be possible to work on both project 2 and 3. Project 1, however, involves some theorems, which will be developed as part of projects 3 and 4. Thus if he chooses to work on project 1, there must be at least 20 hours of work every week on projects 3 and 4. On the other hand, if at least 25 hours a week are used in total on projects 1, 3, 4 and 5, then at least 5 hours should be used on project 2.

Using variables x_i to mean the number of hours used on project i, Professor P7 introduces indicator variables δ_i to indicate whether he should get involved in project i.

Professor P7 would like to maximize the number of projects he is involved in. Thus the natural objective value is

$$\text{maximize } \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5$$

Q 14 Which of the following constraints should be *removed* to get a proper formulation of the problem. (You may assume that x_i is measured in whole hours, i.e. $\epsilon = 1$ in the equations Williams presents in Chapter 9).

14.a) $x_1 + x_2 + x_3 + x_4 + x_5 \leq 37$

14.b) $x_i - 5\delta_i \geq 0$, for $i = 1, \ldots, 5$

14.c) $x_i - 37\delta_i \leq 0$, for $i = 1, \ldots, 5$

14.d) $\delta_2 + \delta_3 \leq 1$

14.e) $x_3 + x_4 - 20\delta_1 \geq 0$

14.f) $x_1 + x_3 + x_4 + x_5 + 26\delta_2 \geq 26$

14.g) $x_1 + x_3 + x_4 + x_5 - 13\delta_2 \leq 24$

14.h) $\delta_i \in \{0, 1\}$, for $i = 1, \ldots, 5$

□

Time is not the only resource for finishing a project. Also money is needed to cover travel expenses, conferences, and computing facilities. Professor P7 has a budget of 20 (thousand kroner) to cover such expenses. Knowing the money needed for each project, he derives the following inequality:

$$5\delta_1 + 8\delta_2 + 4\delta_3 + 9\delta_4 + 6\delta_5 \leq 20$$

Q 15 Some of the indices form a minimal cover (see Williams chapter 10 for a definition). Which set of indices does *not* form a minimal cover. We assume that a_i is the coefficient of δ_i, i.e. $a_1 = 5$, $a_2 = 8$, $a_3 = 4$, $a_4 = 9$, and $a_5 = 6$.

15.a) indices $\{1, 2, 4\}$.

15.b) indices $\{1, 2, 3, 5\}$.

15.c) indices $\{1, 4, 5\}$.

15.d) indices $\{2, 3, 4\}$.

15.e) indices $\{2, 4, 5\}$.

15.f) all the proposed sets form a minimal cover.

□

Q 16 One of the minimal covers can be extended to a strong cover inequality of the form

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 \leq m$$

What is the smallest, correct value of m

16.a) $m = 1$

16.b) $m = 2$

16.c) $m = 3$

16.d) $m = 4$

16.e) $m = 5$

□

Questions Q17-Q20: Touring Medal

An academic career is not complete without having obtained some medals or awards. Thus professor P7 is working hard on getting the *Touring Medal*, which is the equivalent of a Nobel Prize in Combinatorial Optimization. By solving the following very hard problem, professor P7 is almost sure of getting the *Touring Medal* before year 2000.

$$
\begin{array}{llllllll}
\text{max} & x_1 & +x_2 & +x_3 & -x_4 & -x_5 & & \\
\text{s.t.} & x_1 & -x_2 & & & & \leq & 3 \\
 & & x_2 & & & -x_5 & \leq & 6 \\
 & x_1 & +x_2 & +x_3 & & & \leq & 8 \\
 & x_1 & & -x_3 & & & \leq & 7 \\
 & & x_2 & & -x_4 & & \leq & 17 \\
 & \multicolumn{7}{l}{x_i \geq 0, \quad i = 1, \ldots, 5}
\end{array}
$$

The simplex algorithm returns the solution $x_1 = \frac{11}{2}$, $x_2 = \frac{5}{2}$, $x_3 = x_4 = x_5 = 0$. Use complementary slackness to find the dual solution.

Q 17 What is the value of the first dual variable y_1 ?

17.a) $y_1 = 0$

17.b) $y_1 = \frac{11}{2}$

17.c) $y_1 = 4$

17.d) $y_1 = 1$

17.e) $y_1 = \frac{3}{2}$

17.f) $y_1 = 2$

17.g) $y_1 = -1$

□

Professor P7 would like to solve the problem to integer optimality but, as mentioned above, the simplex algorithm returns a solution which is not particularly integral. Thinking about the problem for a couple of days, Professor P7 gets a bright idea: If we remove one of the constraints, the remaining constraint matrix satisfies property P (Williams, Chapter 10) and is thus totally unimodular.

Q 18 For the constraint matrix to satisfy property P, which constraint should be removed?

18.a) $x_1 - x_2 \leq 3$

18.b) $x_2 - x_5 \leq 6$

18.c) $x_1 + x_2 + x_3 \leq 8$

18.d) $x_1 - x_3 \leq 7$

18.e) $x_2 - x_4 \leq 17$

□

Instead of removing the constraint, one can use Lagrangian relaxation to somehow punish violated constraints. Using multiplier $\lambda = 2$ for the constraint answering Q18, a reduced problem with only four constraints results.

Q 19 What is the objective function of the reduced problem

19.a) $-x_1 + 3x_2 + x_3 - x_4 - x_5 + 6$

19.b) $x_1 - x_2 + x_3 - x_4 + x_5 + 12$

19.c) $3x_1 + 3x_2 + 3x_3 - x_4 - x_5 - 16$

19.d) $-x_1 - x_2 - x_3 - x_4 - x_5 + 16$

19.e) $3x_1 + x_2 - x_3 - x_4 - x_5 - 14$

19.f) $x_1 - x_2 + x_3 - 3x_4 - x_5$

19.g) $x_1 - x_2 + x_3 - 3x_4 - x_5 + 34$

□

Since the constraint matrix of the reduced problem is Totally Unimodular, the multiplier λ leading to the tightest upper bound may be found through linear programming.

Q 20 What is the choice of λ leading to the tightest upper bound

20.a) $\lambda = -3$.

20.b) $\lambda = 1$.

20.c) $\lambda = \frac{11}{2}$.

20.d) $\lambda = 7$.

20.e) $\lambda = -1$.

20.f) $\lambda = \frac{5}{2}$.

20.g) $\lambda = 0$.

□