

Friday, December 17

Vigtigste emner

- Totalt unimodulær
- Modelbygning (snydearket)
- Branch-and-bound
- Grænseværdier
- Gomory snit, Chvatal snit, Chvatal-Gomory snit
- Facetter, dimension af løsningsrum
- Cover uligheder, løft af uligheder, deltur-uligheder
- Relaxeringer og deres styrke (Lagrange, Surrogat, LP)
- Dantzig-Wolfe, Kolonnegenerering (se plancherne)

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Bedste forberedelse

Forstå sammenhæng

- Lagrange relaxering og LP-relaxering
- Styrke af Lagrange relaxering
- Lagrange relaxering og Dantzig-Wolfe dekomponering

Modelbygning

- Jo flere modeller I prøver, des lettere
- Lær at bruge snydearket

Plancher

- Læsevejledning
- Eksempler på anvendelser

Gamle eksamenssæt

- Regn nogle stykker
- Kontroller med vejledende løsninger
- Diskuter med hinanden

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Complementary slack sætning

Kan bruges til at komme fra primal løsning til dual løsning.

Enten er en begrænsning bindende, eller også er den tilhørende duale variabel nul

primale problem

$$\begin{aligned} & \text{maximize} && \sum_{j=1}^n c_j x_j \\ & \text{subject to} && \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1, \dots, m \\ & && x_j \geq 0 \quad j = 1, \dots, n \end{aligned}$$

duale problem

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^m b_i y_i \\ & \text{subject to} && \sum_{i=1}^m a_{ij} y_i \geq c_j \quad j = 1, \dots, n \\ & && y_i \geq 0 \quad i = 1, \dots, m \end{aligned}$$

Opgave 1,2 i IP-opgaver, samt ny opgave 26 i IP-opgaver (på hjemmesiden)

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Duality

Branch-and-bound, economics: upper bound on LP.

$$\begin{aligned} & \text{maximize} && 4x_1 + x_2 + 5x_3 + 3x_4 \\ & \text{subject to} && x_1 - x_2 - x_3 + 3x_4 \leq 1 \\ & && 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\ & && -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\ & && x_1, x_2, x_3, x_4 \geq 0 \end{aligned} \quad (1)$$

Multiplying the second constraint by two

$$10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110$$

thus $z^* \leq 110$.

Linear combination of some constraints: second and third constraint

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

thus $z^* \leq 58$.

In general *any* linear combination.

multipliers y_1, y_2, y_3 , demand $y_1, y_2, y_3 \geq 0$

$$\begin{aligned} & y_1(x_1 - x_2 - x_3 + 3x_4) + \\ & y_2(5x_1 + x_2 + 3x_3 + 8x_4) + \\ & y_3(-x_1 + 2x_2 + 3x_3 - 5x_4) \leq y_1 + 55y_2 + 3y_3 \end{aligned}$$

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which is equivalent to

$$\begin{aligned} & (y_1 + 5y_2 - y_3)x_1 + \\ & (-y_1 + y_2 + 2y_3)x_2 + \\ & (-y_1 + 3y_2 + 3y_3)x_3 + \\ & (3y_1 + 8y_2 + 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3 \end{aligned} \quad (2)$$

coefficients must exceed those in (1):

$$\begin{aligned} y_1 + 5y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2y_3 &\geq 1 \\ -y_1 + 3y_2 + 3y_3 &\geq 5 \\ 3y_1 + 8y_2 + 5y_3 &\geq 3 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

minimize the right-hand side of (2).

dual problem:

$$\begin{aligned} \text{minimize} \quad & y_1 + 55y_2 + 3y_3 \\ \text{subject to} \quad & y_1 + 5y_2 - y_3 \geq 4 \\ & -y_1 + y_2 + 2y_3 \geq 1 \\ & -y_1 + 3y_2 + 3y_3 \geq 5 \\ & 3y_1 + 8y_2 + 5y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

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Duality

primal problem.

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^n c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1, \dots, m \\ & x_j \geq 0 \quad j = 1, \dots, n \end{aligned}$$

associated dual problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^m b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^m a_{ij} y_i \geq c_j \quad j = 1, \dots, n \\ & y_i \geq 0 \quad i = 1, \dots, m \end{aligned}$$

Weak duality

For every primal feasible solution (x_1, \dots, x_n)
for every dual feasible solution (y_1, \dots, y_m) :

$$\sum_{j=1}^n c_j x_j \leq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \leq \sum_{i=1}^m b_i y_i$$

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Gomory Cuts - example

Adding slack variables $x_3, x_4, x_5 \geq 0$, and solving LP-problem (Taha simplex table)

basis	z	x_1	x_2	x_3	x_4	x_5	solution
z	1			$\frac{4}{7}$	$\frac{1}{7}$		$\frac{59}{7}$
x_1		1		$\frac{1}{7}$	$\frac{2}{7}$		$\frac{20}{7}$
x_2			1		1		3
x_5				$-\frac{2}{7}$	$\frac{10}{7}$	1	$\frac{23}{7}$

The simplex table as equations

$$A_B x_B + A_N x_N = b$$

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$\begin{aligned} \text{max} \quad & \frac{59}{7} \\ \text{s.t.} \quad & x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7} \\ & x_2 + x_4 = 3 \\ & -\frac{2}{7}x_3 + \frac{10}{7}x_4 + x_5 = \frac{23}{7} \\ & x_1, x_2, x_3, x_4, x_5 \geq 0, \text{ integer} \end{aligned}$$

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The optimal LP-solution is

$$(x_1, x_2, x_3, x_4, x_5) = \left(\frac{20}{7}, 3, 0, 0, \frac{23}{7} \right)$$

which is fractional.

From first equation in Simplex table we get

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7}$$

and hence also $x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 \leq \frac{20}{7}$, so

$$x_1 + \left[\frac{1}{7} \right] x_3 + \left[\frac{2}{7} \right] x_4 \leq \left[\frac{20}{7} \right]$$

inserting $x_1 = -\frac{1}{7}x_3 - \frac{2}{7}x_4 + \frac{20}{7}$ we get

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

or substituting the slack variables x_3 and x_4 we get

$$\frac{1}{7}(14 - 7x_1 + 2x_2) + \frac{2}{7}(3 - x_2) \geq \frac{6}{7}$$

which can be reduced to $x_1 \leq 2$.

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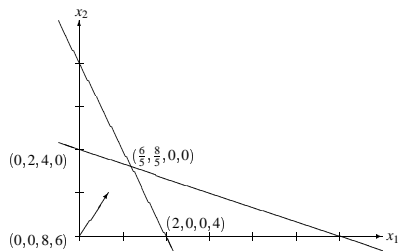
Linear Programming

$$\begin{aligned} & \text{maximize } 2x_1 + 3x_2 \\ & \text{subject to } 4x_1 + 2x_2 \leq 8 \\ & \quad \quad \quad x_1 + 3x_2 \leq 6 \\ & \quad \quad \quad x_1, x_2 \geq 0 \end{aligned}$$

Add slack variables

$$\begin{aligned} & \text{maximize } 2x_1 + 3x_2 \\ & \text{subject to } 4x_1 + 2x_2 + x_3 = 8 \\ & \quad \quad \quad x_1 + 3x_2 + x_4 = 6 \\ & \quad \quad \quad x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The set of constraints form a polyhedral.
Optimal solution is found at extreme points



Extreme points:

$$\begin{aligned} & (0, 0, 8, 6) \quad (0, 4, 0, -6) \quad (0, 2, 4, 0) \\ & (2, 0, 0, 4) \quad (6, 0, -16, 0) \quad \left(\frac{6}{5}, \frac{8}{5}, 0, 0\right) \end{aligned}$$

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Extreme point

- Extreme points appear by setting $n - m$ variables to 0 and solving the remaining m equations with m variables to optimality.
- Choose m linearly independent columns in A . The corresponding set $B = \{i_1, i_2, \dots, i_m\}$ is called a *basis*.
- A simple algorithm: Search through all extreme points
Basis can be chosen in $\binom{n}{m}$ ways (i.e. exponential).
- Two basis feasible solutions x^1 and x^2 are adjacent if B^1 and B^2 have $m - 1$ common elements.
- *Simplex algorithm* is a greedy algorithm which works as follows: Move from basis feasible solution to adjacent basis feasible solution such that objective function is "increased most possible" in each step.
 - Initial solution
 - Iterative step
 - Optimality criteria

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Simplex in Matrix Form (Taha Chapter 6)

LP-model

$$\begin{aligned} & \text{maximize } cx \\ & \text{subject to } Ax \leq b \\ & \quad \quad \quad x \geq 0 \end{aligned}$$

Formulation after adding slack variables (new A, c, x)

$$\begin{aligned} & \text{maximize } cx \\ & \text{subject to } Ax = b \\ & \quad \quad \quad x \geq 0 \end{aligned}$$

If we have m constraints, the Simplex algorithm chooses m linearly independent columns in A (the basis). The corresponding variables are x_B the remaining variables x_N

$$\begin{aligned} & \text{maximize } c_B x_B + c_N x_N \\ & \text{subject to } A_B x_B + A_N x_N = b \\ & \quad \quad \quad x_B, x_N \geq 0 \end{aligned}$$

Solve for x_B

$$x_B = A_B^{-1}(b - A_N x_N)$$

setting the non-basis variables to zero $x_N = 0$ we get

$$x_B = A_B^{-1}b$$

which is a basis solution. Objective value

$$c_B x_B = c_B A_B^{-1}b$$

objective function

$$z = c_B A_B^{-1}b + x_N(c_N - c_B A_B^{-1}A_N)$$

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Since dual variables $y = c_B A_B^{-1}$ when simplex terminates

$$z = yb + x_N(c_N - yA_N)$$

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Simplex in matrix form (Taha section 7.1)

$$\begin{aligned} &\text{maximize } cx \\ &\text{subject to } Ax = b \\ & \quad x \geq 0, \end{aligned}$$

Reformulation in matrix form

$$\begin{pmatrix} 1 & -c \\ 0 & A \end{pmatrix} \begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Assume

- x_B set of basis variables
- c_B coefficients in c corresponding to basis
- A_B coefficients in A corresponding to basis

By multiplying with

$$\begin{pmatrix} 1 & c_B A_B^{-1} \\ 0 & A_B^{-1} \end{pmatrix}$$

get equivalent form

$$\begin{pmatrix} 1 & c_B A_B^{-1} A - c \\ 0 & A_B^{-1} A \end{pmatrix} \begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} c_B A_B^{-1} b \\ A_B^{-1} b \end{pmatrix}$$

Table page 295 in Taha

basis	x_j	solution
z	$c_B A_B^{-1} A_j - c_j$	$c_B A_B^{-1} b$
x_B	$A_B^{-1} A_j$	$A_B^{-1} b$

Simplex in matrix form

Basis variables and non-basis variables

$$x_B = (x_1, x_2) \quad x_N = (x_3, x_4, x_5, \dots)$$

split matrix

$$A_B = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix} \quad A_N = \begin{pmatrix} 1 & 2 & 3 & \dots \\ 5 & 4 & 2 & \dots \end{pmatrix}$$

reformulated problem

$$\begin{aligned} &\min \quad c_B x_B + c_N x_N \\ &\text{s.t.} \quad A_B x_B + A_N x_N = b \\ & \quad x \geq 0 \end{aligned}$$

Simplex algorithm sets $x_N = 0$ and solves $A_B x_B = b$ getting

$$x_B = A_B^{-1} b$$

corresponding objective function

$$z = c_B A_B^{-1} b + x_N (c_N - c_B A_B^{-1} A_N)$$

Since dual variables $y = c_B A_B^{-1}$ we have

$$z = y b + x_N (c_N - y A_N)$$