Friday, December 17

Vigtigste emner

- Totalt unimodulær
- Modelbygning (snydearket)
- Branch-and-bound
- Grænseværdier
- Gomory snit, Chvatal snit, Chvatal-Gomory snit
- Facetter, dimension af løsningsrum
- Cover uligheder, løft af uligheder, deltur-uligheder
- Relaxeringer og deres styrke (Lagrange, Surrogat, LP)
- Dantzig-Wolfe, Kolonnegenerering (se plancherne)

Bedste forberedelse

Forstå sammenhæng

- Lagrange relaxering og LP-relaxering
- Styrke af Lagrange relaxering
- Lagrange relaxering og Dantzig-Wolfe dekomponering

Modelbygning

- Jo flere modeller I prøver, des lettere
- Lær at bruge snydearket

Plancher

- Læsevejledning
- Eksempler på anvendelser

Gamle eksamenssæt

- Regn nogle stykker
- Kontroller med vejledende løsninger
- Diskuter med hinanden

Complementary slack sætning

Kan bruges til at komme fra primal løsning til dual løsning.

Enten er en begrænsning bindende, eller også er den tilhørende duale variabel nul

primale problem

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m \\ & x_j \geq 0 \quad j = 1, \ldots, n \end{aligned}$$

duale problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_j y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad j = 1, \ldots, n \\ & y_i \geq 0 \quad i = 1, \ldots, m \end{aligned}$$

Opgave 1,2 i IP-opgaver, samt ny opgave 26 i IP-opgaver (på hjemmesiden)

Duality

Branch-and-bound, economics: upper bound on LP.

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array} \tag{1}$$

Multiplying the second constraint by two

$$10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110$$

thus $z^* \leq 110$.
Linear combination of some constraints: second and third constraint

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

thus $z^* \leq 58$.
In general *any* linear combination.
multipliers y_1, y_2, y_3, demand $y_1, y_2, y_3 \geq 0$

$$\begin{array}{l}
y_1(x_1 - x_2 - x_3 + 3x_4) + \\
y_2(5x_1 + x_2 + 3x_3 + 8x_4) + \\
y_3(-x_1 + 2x_2 + 3x_3 - 5x_4) \leq y_1 + 55y_2 + 3y_3
\end{array}$$

which is equivalent to

$$
\begin{array}{l}
(y_1 + 5y_2 - y_3)x_1 + \\
(-y_1 + y_2 + 2y_3)x_2 + \\
(-y_1 + 3y_2 + 3y_3)x_3 + \\
(3y_1 + 8y_2 + 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3
\end{array}
\tag{2}
$$

coefficients must exceed those in (1):

$$
\begin{array}{rcrcrcl}
y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
-y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
-y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
3y_1 & + & 8y_2 & + & 5y_3 & \geq & 3 \\
 & & & & y_1, y_2, y_3 & \geq & 0
\end{array}
$$

minimize the right-hand side of (2).
dual problem:

$$
\begin{array}{lrcrcrcl}
\text{minimize} & y_1 & + & 55y_2 & + & 3y_3 & & \\
\text{subject to} & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
 & -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
 & 3y_1 & + & 8y_2 & + & 5y_3 & \geq & 3 \\
 & & & & & y_1, y_2, y_3 & \geq & 0
\end{array}
$$

Duality

primal problem.

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, \dots, m \\ & x_j \ge 0 \quad j = 1, \dots, n \end{aligned}$$

associated dual problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_j y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad j = 1, \dots, n \\ & y_i \ge 0 \quad i = 1, \dots, m \end{aligned}$$

Weak duality

For every primal feasible solution $(x_1, \dots, x_n)$
for every dual feasible solution $(y_1, \dots, y_m)$:

$$\sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \le \sum_{i=1}^{m} b_i y_i$$

Gomory Cuts - example

Adding slack variables $x_3, x_4, x_5 \geq 0$, and solving LP-problem (Taha simplex table)

basis	z	x_1	x_2	x_3	x_4	x_5	solution
z	1			$\frac{4}{7}$	$\frac{1}{7}$		$\frac{59}{7}$
x_1		1		$\frac{1}{7}$	$\frac{2}{7}$		$\frac{20}{7}$
x_2			1		1		3
x_5				$-\frac{2}{7}$	$\frac{10}{7}$	1	$\frac{23}{7}$

The simplex table as equations

$$A_B x_B + A_N x_N = b$$

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$\begin{array}{llllllllll}
\max & \frac{59}{7} & & & - \frac{4}{7}x_3 & - \frac{1}{7}x_4 & & & \\
\text{s.t.} & & x_1 & & + \frac{1}{7}x_3 & + \frac{2}{7}x_4 & & = & \frac{20}{7} \\
& & & x_2 & & + x_4 & & = & 3 \\
& & & & - \frac{2}{7}x_3 & + \frac{10}{7}x_4 & + x_5 & = & \frac{23}{7} \\
& & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0, \text{ integer}
\end{array}$$

The optimal LP-solution is

$$(x_1, x_2, x_3, x_4, x_5) = (\frac{20}{7}, 3, 0, 0, \frac{23}{7})$$

which is fractional.

From first equation in Simplex table we get

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7}$$

and hence also $x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 \leq \frac{20}{7}$, so

$$x_1 + \left\lfloor \frac{1}{7} \right\rfloor x_3 + \left\lfloor \frac{2}{7} \right\rfloor x_4 \leq \left\lfloor \frac{20}{7} \right\rfloor$$

inserting $x_1 = -\frac{1}{7}x_3 - \frac{2}{7}x_4 + \frac{20}{7}$ we get

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

or substituting the slack variables x_3 and x_4 we get

$$\frac{1}{7}(14 - 7x_1 + 2x_2) + \frac{2}{7}(3 - x_2) \geq \frac{6}{7}$$

which can be reduced to $x_1 \leq 2$.

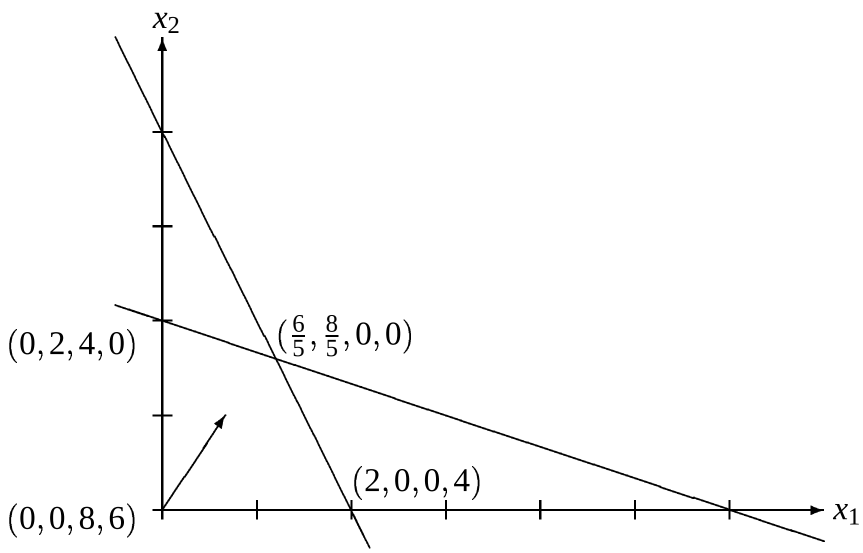
Linear Programming

$$\begin{array}{lrcrcl} \text{maximize} & 2x_1 & + & 3x_2 & & \\ \text{subject to} & 4x_1 & + & 2x_2 & \leq & 8 \\ & x_1 & + & 3x_2 & \leq & 6 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Add slack variables

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 2x_1 & + & 3x_2 & & & & & & \\ \text{subject to} & 4x_1 & + & 2x_2 & + & x_3 & & & = & 8 \\ & x_1 & + & 3x_2 & & & + & x_4 & = & 6 \\ & & & & & & x_1, x_2, & x_3, x_4 & \geq & 0 \end{array}$$

The set of constraints form a polyhedral.
Optimal solution is found at extreme points



Extreme points:

$$\begin{array}{ccc} (0,0,8,6) & (0,4,0,-6) & (0,2,4,0) \\ (2,0,0,4) & (6,0,-16,0) & (\frac{6}{5},\frac{8}{5},0,0) \end{array}$$

Extreme point

- Extreme points appear by setting $n-m$ variables to 0 and solving the remaining m equations with m variables to optimality.
- Choose m linearly independant columns in A. The corresponding set $B = \{i_1, i_2, \ldots, i_m\}$ is called a *basis*.
- A simple algorithm: Search through all extreme points

  Basis can be chosen in $\begin{pmatrix} n \\ m \end{pmatrix}$ ways (i.e. exponential).

- Two basis feasible solutions x^1 and x^2 are adjacent if B^1 and B^2 have $m-1$ common elements.
- *Simplex algorithm* is a greedy algorithm which works as follows: Move from basis feasible solution to adjacent basis feasible solution such that objective function is ”increased most possible” in each step.
  - Initial solution
  - Iterative step
  - Optimality criteria

Simplex in Matrix Form (Taha Chapter 6)

LP-model

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

Formulation after adding slack variables (new A, c, x)

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

If we have m constraints, the Simplex algorithm chooses m linearly independant columns in A (the basis). The corresponding variables are x_B the remaining variables x_N

$$\begin{array}{ll} \text{maximize} & c_B x_B + c_N x_N \\ \text{subject to} & A_B x_B + A_N x_N = b \\ & x_B, x_N \geq 0 \end{array}$$

Solve for x_B

$$x_B = A_B^{-1}(b - A_N x_N)$$

setting the non-basis variables to zero $x_N = 0$ we get

$$x_B = A_B^{-1} b$$

which is a basis solution. Objective value

$$c_B x_B = c_B A_B^{-1} b$$

objective function

$$z = c_B A_B^{-1} b + x_N(c_N - c_B A_B^{-1} A_N)$$

Since dual variables $y = c_B A_B^{-1}$ when simplex terminates

$$z = yb + x_N(c_N - yA_N)$$

Simplex in matrix form (Taha section 7.1)

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array}$$

Reformulation in matrix form

$$\begin{pmatrix} 1 & -c \\ 0 & A \end{pmatrix} \begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Assume

- x_B set of basis variables
- c_B coefficients in c corresponding to basis
- A_B coefficients in A corresponding to basis

By multiplying with

$$\begin{pmatrix} 1 & c_B A_B^{-1} \\ 0 & A_B^{-1} \end{pmatrix}$$

get equivalent form

$$\begin{pmatrix} 1 & c_B A_B^{-1} A - c \\ 0 & A_B^{-1} A \end{pmatrix} \begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} c_B A_B^{-1} b \\ A_B^{-1} b \end{pmatrix}$$

Table page 295 in Taha

basis	x_j	solution
z	$c_B A_B^{-1} A_j - c_j$	$c_B A_B^{-1} b$
x_B	$A_B^{-1} A_j$	$A_B^{-1} b$

Simplex in matrix form

Basis variables and non-basis variables

$$x_B = (x_1, x_2) \qquad\qquad x_N = (x_3, x_4, x_5, \cdots)$$

split matrix

$$A_B = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix} \qquad\qquad A_N = \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 5 & 4 & 2 & \cdots \end{pmatrix}$$

reformulated problem

$$\begin{array}{ll} \min & c_B x_B + c_N x_N \\ \text{s.t.} & A_B x_B + A_N x_N = b \\ & x \geq 0 \end{array}$$

Simplex algorithm sets $x_N = 0$ and solves $A_B x_B = b$ getting

$$x_B = A_B^{-1} b$$

corresponding objective function

$$z = c_B A_B^{-1} b + x_N (c_N - c_B A_B^{-1} A_N)$$

Since dual variables $y = c_B A_B^{-1}$ we have

$$z = yb + x_N (c_N - y A_N)$$