## Friday, December 10

Program of the day:

- TSP konkurrence
- Delayed column generation — more formally
- Branch-and-price algorithms
- Example: VRP problems solved through delayed column generation
- Evaluation



## Dantzig-Wolfe Decomposition

The problem is split into a master problem and a subproblem

- Tighter bounds
- Better controle of subproblem
- Model may become (very) large

## Delayed column generation

Write up the decomposed model gradually as needed

- Generate a few solutions to the subproblems
- Solve the master problem to LP-optimality
- Use the dual information to find most promising solutions to the subproblem
- Extend the master problem with the new subproblem solutions.



## Decomposition

If model has "block" structure

$$\begin{array}{lccccccc}
\max & c^1x^1 & + & c^2x^2 & + \ldots + & c^Kx^K & & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^Kx^K & = & b \\
& D^1x^1 & + & & & & \leq & d_1 \\
& & + & D^2x^2 & & & \leq & d_2 \\
& & & & \ldots & & \leq & \vdots \\
& & & & & D^Kx^K & \leq & d_K \\
& x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}$$

## Lagrangian relaxation

Objective becomes

$$c^1x^1 + c^2x^2 + \ldots + c^Kx^K$$
$$-\lambda\left(A^1x^1 + A^2x^2 + \ldots + A^Kx^K - b\right)$$

Decomposed into

$$\begin{array}{lccccccc}
\max & c^1x^1 - \lambda A^1x^1 & + & c^2x^2 - \lambda A^2x^2 & + \ldots + & c^Kx^K - \lambda A^Kx^K & + & b \\
\text{s.t.} & D^1x^1 & + & & & & \leq & d_1 \\
& & + & D^2x^2 & & & \leq & d_2 \\
& & & & \ldots & & \leq & \vdots \\
& & & & & D^Kx^K & \leq & d_K \\
& x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}$$

Model is separable



## Dantzig-Wolfe decomposition

If model has "block" structure

$$\begin{array}{lccccccc}
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\text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^Kx^K & = & b \\
& D^1x^1 & + & & & & \leq & d_1 \\
& & + & D^2x^2 & & & \leq & d_2 \\
& & & & \ldots & & \leq & \vdots \\
& & & & & D^Kx^K & \leq & d_K \\
& x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}$$

Substituting each set $X^k$, $k = 1, \ldots, K$ in original model getting *Master Problem*

$$\max c^1\Big(\sum_{t \in T_1} \lambda_{1,t} x^{1,t}\Big) + c^2\Big(\sum_{t \in T_2} \lambda_{2,t} x^{2,t}\Big) + \ldots + c^K\Big(\sum_{t \in T_K} \lambda_{K,t} x^{K,t}\Big)$$

$$\text{s.t. } A^1\Big(\sum_{t \in T_1} \lambda_{1,t} x^{1,t}\Big) + A^2\Big(\sum_{t \in T_2} \lambda_{2,t} x^{2,t}\Big) + \ldots + A^K\Big(\sum_{t \in T_K} \lambda_{K,t} x^{K,t}\Big) = b$$

$$\sum_{t \in T_k} \lambda_{k,t} = 1 \qquad k = 1, \ldots, K$$

$$\lambda_{k,t} \in \{0,1\}, \qquad t \in T_k \;\; k = 1, \ldots, K$$

## Strength of linear master model

Solving LP-relaxation of master problem, is equivalent to (Wolsey Prop 11.1)

$$\begin{array}{lllll} \max & c^1x^1 & + \; c^2x^2 & + \ldots + & c^kx^k & \\ \text{s.t.} & A^1x^1 & + \; A^2x^2 & + \ldots + & A^kx^k & = b \\ & x^1 \in \text{conv}(X^1) & x^2 \in \text{conv}(X^2) & \ldots & x^k \in \text{conv}(X^k) & \end{array}$$

## Strength of Lagrangian relaxation

- $z^{LPM}$ be LP-solution value of master problem
- $z^{LD}$ be solution value of lagrangian dual problem

(Theorem 11.2)

$$z^{LPM} = z^{LD}$$



## Delayed column generation, linear master

(*minimization problem*)

Run Simplex algorithm as if complete master problem was known

- Start with a basis solution
- Solve

  $$x_B = A_B^{-1}b$$

  and find dual variables

  $$y = c_B A_B^{-1}$$

- When choosing entering variable solve pricing problem which minimizes reduced costs

  $$c_j^r = c_j - yA_j$$

- If $c_j^r < 0$ add corresponding column $A_j$ to model and repeat
- If $c_j^r \geq 0$ stop



## Cutting Stock Problem

(*minimization problem*)

$a_{ij}$ is number of pieces of type $i$ cut from pattern $j$
Master problem,

$$\begin{array}{ll} \min & \sum_{j=1}^n u_j \\ \text{s.t.} & \sum_{j=1}^n a_{ij}u_j \geq b_i \quad i = 1,\ldots,m, j = 1,\ldots,n \\ & u_j \in \mathbb{Z}_+ \end{array}$$

Solving linear master through delayed column generation

- Start with patterns which only contain one type $i$
- Solve restricted master
- Dual variables $y_i$ say how "attractive" a type $i$ is
- Pricing problem

$$\begin{aligned} z^S = \min \quad & 1 - \sum_{i=1}^m y_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^m w_i x_i \leq L \\ & x \geq 0, \text{integer} \end{aligned}$$

- stop if $z^S \geq 0$



## Terminology

- Master Problem
- Restricted Master Problem
- Subproblem or Pricing Problem
- Branch and cut:
  Branch-and-bound algorithm using cuts to strengthen bounds.
- Branch and price:
  Branch-and-bound algorithm using column generation to derive bounds.
- One says that discarded columns are "priced out".

## Branch-and-price

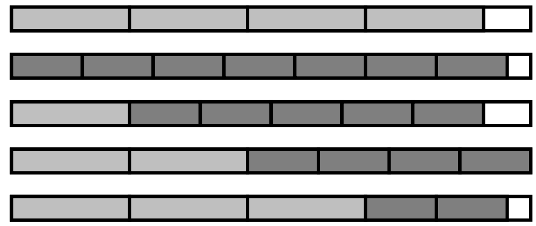
- LP-solution of master problem may have fractional solutions
- Branch-and-bound for getting IP-solution
- In each node solve LP-relaxation of master
- Subproblem may change when we add constraints to master problem
- Branching strategy should make subproblem easy to solve



## Branch-and-price, example

The matrix $A$ contains all different cutting patterns

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 7 & 5 & 4 & 2 \end{pmatrix}$$



Problem

$$\begin{array}{ll} \text{minimize} & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \text{subject to} & 4\lambda_1 + 0\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 \geq 7 \\ & 0\lambda_1 + 7\lambda_2 + 5\lambda_3 + 4\lambda_4 + 2\lambda_5 \geq 3 \\ & \lambda_j \in \mathbb{Z}_+ \end{array}$$

LP-solution $\lambda_1 = 1.375, \lambda_4 = 0.75$

Branch on $\lambda_1 = 0$, $\lambda_1 = 1$, $\lambda_1 = 2$

- Column generation may not generate pattern (4,0)
- Pricing problem is knapsack problem with pattern forbidden



## Branch-and-price, example

Better branching strategy

- Branch 1: item $i$ and item $j$ are cut from *same* pattern
- Branch 2: item $i$ and item $j$ are cut from *different* patterns

Pricing problem

- Branch 1: “glue together” the two items
- Branch 2: multiple-choice knapsack problem

Will the branching terminate?

- Sooner or later we will have defined exactly which items go into same pattern, i.e. all patterns



## Tailing off effect

Column generation may converge slowly in the end

- We do not need exact solution, just lower bound
- Solving master problem for subset of columns does not give valid lower bound (why?)
- Instead we may use Lagrangian relaxation of joint constraint
- “guess” lagrangian multipliers equal to dual variables from master problem

## Heuristic solution

- Restricted master problem will only contain a subset of the columns
- We may solve restricted master problem to IP-optimality
- Restricted master is a “set-covering-like” problem which is not too difficult to solve

## Vehicle Routing Problem with Time Windows

A delivery company shall visit 5 customers in some pre-described time-intervals $[a_i, b_i]$ as follows:

| customer $i$ | $a_i$ | $b_i$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 4 |
| 3 | 1 | 2 |
| 4 | 3 | 4 |
| 5 | 2 | 3 |

The depot is considered as node 0 and 6. The travel time $t_{ij}$ between customer $i$ and $j$, and corresponding cost $c_{ij}$ is

$$c_{ij} = t_{ij} = |j - i| \quad i = 1, \ldots, 6, \; j = 0, \ldots, 6$$

There are at most five vehicles available, and the cost of a vehicle is 6 units, i.e. $c_{0j} = 6$. The travel time from the depot to the first customer is $t_{0j} = 1$.

1 Find a feasible initial solution to the problem.

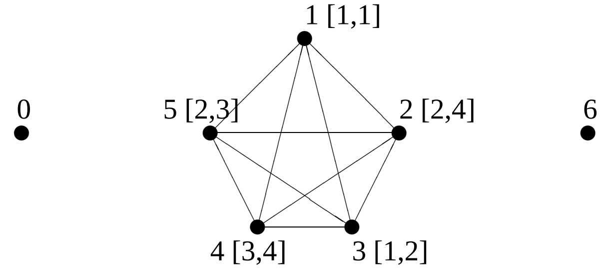
2 Use column generation to solve the problem to LP-optimality.

3 Write up all feasible routes, and solve the problem using CPLEX to integer optimality.

4 Compare the LP solution with the integer solution.



## Graph




$$c_{ij} = t_{ij} = |j - i| \qquad c_{0j} = 6, \; t_{0j} = 1$$

Possible tours (in lexicographic order, dominated removed)

| tour | cost |
|---|---|
| 0,1,2,4,6 | 11 |
| 0,(wait),2,4,6 | 10 |
| 0,3,2,4,6 | 11 |
| 0,3,(wait),4,6 | 10 |
| 0,3,5,6 | 9 |
| 0,3,5,4,6 | 11 |
| 0,(wait),(wait),4,6 | 8 |
| 0,(wait),5,4,6 | 10 |
| 0,(wait),5,6 | 7 |



## Column Generation

- **Initial Solution**. We can choose the trivial routes:

| tour | cost |
|---|---|
| 0,1,6 | 11 |
| 0,2,6 | 10 |
| 0,3,6 | 9 |
| 0,4,6 | 8 |
| 0,5,6 | 7 |

which gives a solution of 45.

- **Column generation**. We start with formulation

$$\begin{array}{lllllll}
\min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & \\
\text{s.t.} & x_1 & & & & & \geq 1 \\
& & x_2 & & & & \geq 1 \\
& & & x_3 & & & \geq 1 \\
& & & & x_4 & & \geq 1 \\
& & & & & x_5 & \geq 1
\end{array}$$

the dual variables corresponding to the constraints are

$$y_1 = 11, \quad y_2 = 10, \quad y_3 = 9, \quad y_4 = 8, \quad y_5 = 7$$

The reduced cost of a tour is now the "original" cost of the edges minus the dual variables corresponding to the customers visited. The most beneficial tour is $(0, 1, 2, 4, 6)$ having the cost $11 - (11 + 10 + 8) = -18$. Since the reduced cost of the new column is



negative, we add it to the problem, getting:

$$\begin{array}{llllllll}
\min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & \\
\text{s.t.} & x_1 & & & & & +x_6 & \geq 1 \\
& & x_2 & & & & +x_6 & \geq 1 \\
& & & x_3 & & & & \geq 1 \\
& & & & x_4 & & +x_6 & \geq 1 \\
& & & & & x_5 & & \geq 1
\end{array}$$

the dual variables are

$$y_1 = 0, \quad y_2 = 3, \quad y_3 = 9, \quad y_4 = 8, \quad y_5 = 7$$

The most beneficial tour is $(0, 3, 5, 4, 6)$ having the reduced cost $11 - (9 + 8 + 7) = -13$, which is added to the problem:

$$\begin{array}{lllllllll}
\min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & +11x_7 & \\
\text{s.t.} & x_1 & & & & & +x_6 & & \geq 1 \\
& & x_2 & & & & +x_6 & & \geq 1 \\
& & & x_3 & & & & +x_7 & \geq 1 \\
& & & & x_4 & & +x_6 & +x_7 & \geq 1 \\
& & & & & x_5 & & +x_7 & \geq 1
\end{array}$$

the dual variables are

$$y_1 = 1, \quad y_2 = 10, \quad y_3 = 4, \quad y_4 = 0, \quad y_5 = 7$$

Adding the tour $(0, 3, 2, 4, 6)$ at reduced cost $11 - (10 + 4 + 0) = -3$ gives

$$\begin{array}{llllllllll}
\min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & +11x_7 & +11x_8 & \\
\text{s.t.} & x_1 & & & & & +x_6 & & & \geq 1 \\
& & x_2 & & & & +x_6 & & +x_8 & \geq 1 \\
& & & x_3 & & & & +x_7 & +x_8 & \geq 1 \\
& & & & x_4 & & +x_6 & +x_7 & +x_8 & \geq 1 \\
& & & & & x_5 & & +x_7 & & \geq 1
\end{array}$$

with dual variables

$$y_1 = 4, \quad y_2 = 7, \quad y_3 = 4, \quad y_4 = 0, \quad y_5 = 7$$

The most beneficial tour is $(0,3,5,6)$, having reduced cost $9-(4+7) = -2$. Thus

$$\begin{array}{lllllllllll}
\min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & +11x_7 & +11x_8 & +9x_9 & \\
\text{s.t.} & x_1 & & & & & +x_6 & & & & \geq 1 \\
& & x_2 & & & & +x_6 & & +x_8 & & \geq 1 \\
& & & x_3 & & & & +x_7 & +x_8 & +x_9 & \geq 1 \\
& & & & x_4 & & +x_6 & +x_7 & +x_8 & & \geq 1 \\
& & & & & x_5 & & +x_7 & & +x_9 & \geq 1
\end{array}$$

with dual variables

$$y_1 = 2, \quad y_2 = 9, \quad y_3 = 2, \quad y_4 = 0, \quad y_5 = 7$$

Now, all tours have reduced costs $\geq 0$, so we are done. The bound found by column generation is thus $z_{LP} = 20$.

- **Write up all feasible solutions**. The problem with all 9 tours added to the formulation, solved to IP-optimality, gives the solution $z_{IP} = 20$.
- **Compare bounds**. We have $z_{IP} = z_{LP}$ so the bounds are tight.