

Friday, December 10

Program of the day:

- TSP konkurrence
- Delayed column generation — more formally
- Branch-and-price algorithms
- Example: VRP problems solved through delayed column generation
- Evaluation

Dantzig-Wolfe Decomposition

The problem is split into a master problem and a subproblem

- Tighter bounds
- Better control of subproblem
- Model may become (very) large

Delayed column generation

Write up the decomposed model gradually as needed

- Generate a few solutions to the subproblems
- Solve the master problem to LP-optimality
- Use the dual information to find most promising solutions to the subproblem
- Extend the master problem with the new subproblem solutions.

Decomposition

If model has “block” structure

$$\begin{array}{llllll}
 \max & c^1 x^1 & + & c^2 x^2 & + \dots + & c^K x^K \\
 \text{s.t.} & A^1 x^1 & + & A^2 x^2 & + \dots + & A^K x^K = b \\
 & D^1 x^1 & + & & & \leq d_1 \\
 & & + & D^2 x^2 & & \leq d_2 \\
 & & & & \dots & \leq \vdots \\
 & & & & & D^K x^K \leq d_K \\
 & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \dots & x^K \in \mathbb{Z}_+^{n_K}
 \end{array}$$

Lagrangian relaxation

Objective becomes

$$\begin{aligned}
 & c^1 x^1 + c^2 x^2 + \dots + c^K x^K \\
 & - \lambda (A^1 x^1 + A^2 x^2 + \dots + A^K x^K - b)
 \end{aligned}$$

Decomposed into

$$\begin{array}{llllll}
 \max & c^1 x^1 - \lambda A^1 x^1 & + & c^2 x^2 - \lambda A^2 x^2 & + \dots + & c^K x^K - \lambda A^K x^K + b \\
 \text{s.t.} & D^1 x^1 & + & & & \leq d_1 \\
 & & + & D^2 x^2 & & \leq d_2 \\
 & & & & \dots & \leq \vdots \\
 & & & & & D^K x^K \leq d_K \\
 & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \dots & x^K \in \mathbb{Z}_+^{n_K}
 \end{array}$$

Model is separable

Dantzig-Wolfe decomposition

If model has “block” structure

$$\begin{array}{llllll}
 \max & c^1 x^1 & + & c^2 x^2 & + \dots + & c^K x^K \\
 \text{s.t.} & A^1 x^1 & + & A^2 x^2 & + \dots + & A^K x^K & = & b \\
 & D^1 x^1 & + & & & & \leq & d_1 \\
 & & + & D^2 x^2 & & & \leq & d_2 \\
 & & & & \dots & & \leq & \vdots \\
 & & & & & D^K x^K & \leq & d_K \\
 & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \dots & x^K \in \mathbb{Z}_+^{n_K} & &
 \end{array}$$

Substituting each set X^k , $k = 1, \dots, K$ in original model getting *Master Problem*

$$\begin{array}{ll}
 \max & c^1 \left(\sum_{t \in T_1} \lambda_{1,t} x^{1,t} \right) + c^2 \left(\sum_{t \in T_2} \lambda_{2,t} x^{2,t} \right) + \dots + c^K \left(\sum_{t \in T_K} \lambda_{K,t} x^{K,t} \right) \\
 \text{s.t.} & A^1 \left(\sum_{t \in T_1} \lambda_{1,t} x^{1,t} \right) + A^2 \left(\sum_{t \in T_2} \lambda_{2,t} x^{2,t} \right) + \dots + A^K \left(\sum_{t \in T_K} \lambda_{K,t} x^{K,t} \right) = b \\
 & \sum_{t \in T_k} \lambda_{k,t} = 1 \quad k = 1, \dots, K \\
 & \lambda_{k,t} \in \{0, 1\}, \quad t \in T_k \quad k = 1, \dots, K
 \end{array}$$

Strength of linear master model

Solving LP-relaxation of master problem, is equivalent to (Wolsey Prop 11.1)

$$\begin{array}{llllll} \max & c^1 x^1 & + & c^2 x^2 & + \dots + & c^k x^k \\ \text{s.t.} & A^1 x^1 & + & A^2 x^2 & + \dots + & A^k x^k & = b \\ & x^1 \in \text{conv}(X^1) & & x^2 \in \text{conv}(X^2) & \dots & x^k \in \text{conv}(X^k) & \end{array}$$

Strength of Lagrangian relaxation

- z^{LPM} be LP-solution value of master problem
- z^{LD} be solution value of lagrangian dual problem

(Theorem 11.2)

$$z^{LPM} = z^{LD}$$

Delayed column generation, linear master

(*minimization problem*)

Run Simplex algorithm as if complete master problem was known

- Start with a basis solution
- Solve

$$x_B = A_B^{-1}b$$

and find dual variables

$$y = c_B A_B^{-1}$$

- When choosing entering variable solve pricing problem which minimizes reduced costs

$$c_j^r = c_j - yA_j$$

- If $c_j^r < 0$ add corresponding column A_j to model and repeat
- If $c_j^r \geq 0$ stop

Cutting Stock Problem

(*minimization problem*)

a_{ij} is number of pieces of type i cut from pattern j
Master problem,

$$\begin{aligned} \min \quad & \sum_{j=1}^n u_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} u_j \geq b_i \quad i = 1, \dots, m, j = 1, \dots, n \\ & u_j \in \mathbb{Z}_+ \end{aligned}$$

Solving linear master through delayed column generation

- Start with patterns which only contain one type i
- Solve restricted master
- Dual variables y_i say how “attractive” a type i is
- Pricing problem

$$\begin{aligned} z^S = \min \quad & 1 - \sum_{i=1}^m y_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^m w_i x_i \leq L \\ & x \geq 0, \text{ integer} \end{aligned}$$

- stop if $z^S \geq 0$

Terminology

- Master Problem
- Restricted Master Problem
- Subproblem or Pricing Problem
- Branch and cut:
Branch-and-bound algorithm using cuts to strengthen bounds.
- Branch and price:
Branch-and-bound algorithm using column generation to derive bounds.
- One says that discarded columns are “priced out”.

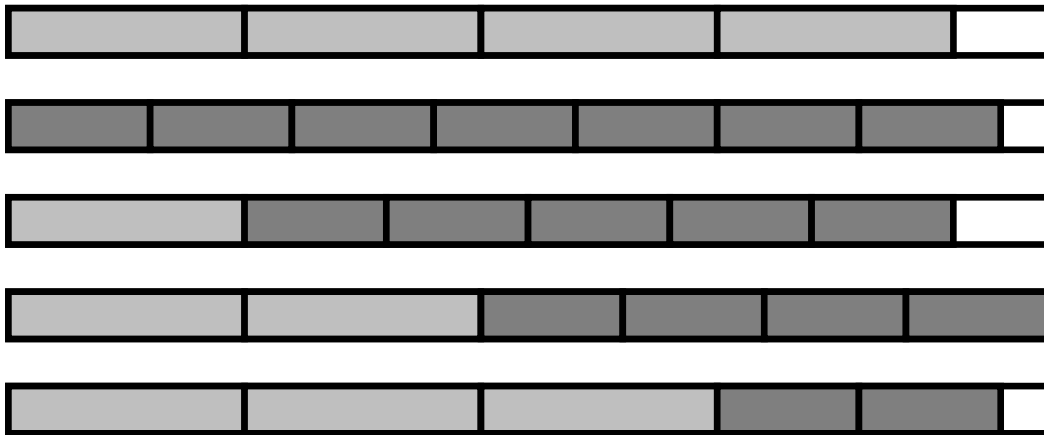
Branch-and-price

- LP-solution of master problem may have fractional solutions
- Branch-and-bound for getting IP-solution
- In each node solve LP-relaxation of master
- Subproblem may change when we add constraints to master problem
- Branching strategy should make subproblem easy to solve

Branch-and-price, example

The matrix A contains all different cutting patterns

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 7 & 5 & 4 & 2 \end{pmatrix}$$



Problem

$$\text{minimize } \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$$

$$\text{subject to } 4\lambda_1 + 0\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 \geq 7$$

$$0\lambda_1 + 7\lambda_2 + 5\lambda_3 + 4\lambda_4 + 2\lambda_5 \geq 3$$

$$\lambda_j \in \mathbb{Z}_+$$

LP-solution $\lambda_1 = 1.375, \lambda_4 = 0.75$

Branch on $\lambda_1 = 0, \lambda_1 = 1, \lambda_1 = 2$

- Column generation may not generate pattern (4,0)
- Pricing problem is knapsack problem with pattern forbidden

Branch-and-price, example

Better branching strategy

- Branch 1: item i and item j are cut from *same* pattern
- Branch 2: item i and item j are cut from *different* patterns

Pricing problem

- Branch 1: “glue together” the two items
- Branch 2: multiple-choice knapsack problem

Will the branching terminate?

- Sooner or later we will have defined exactly which items go into same pattern, i.e. all patterns

Tailing off effect

Column generation may converge slowly in the end

- We do not need exact solution, just lower bound
- Solving master problem for subset of columns does not give valid lower bound (why?)
- Instead we may use Lagrangian relaxation of joint constraint
- “guess” lagrangian multipliers equal to dual variables from master problem

Heuristic solution

- Restricted master problem will only contain a subset of the columns
- We may solve restricted master problem to IP-optimality
- Restricted master is a “set-covering-like” problem which is not too difficult to solve

Vehicle Routing Problem with Time Windows

A delivery company shall visit 5 customers in some pre-described time-intervals $[a_i, b_i]$ as follows:

customer i	a_i	b_i
1	1	1
2	2	4
3	1	2
4	3	4
5	2	3

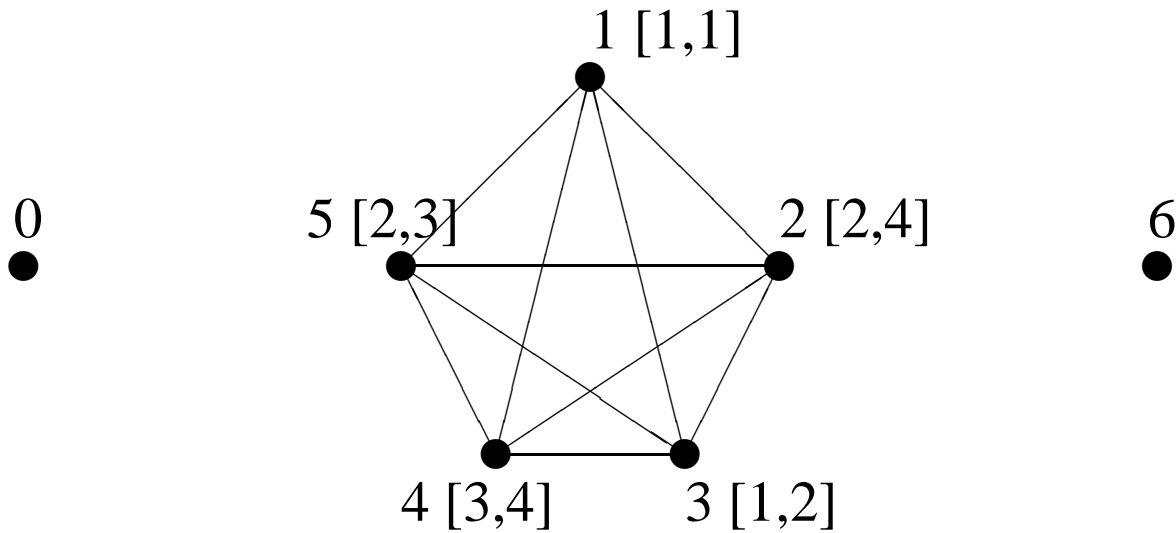
The depot is considered as node 0 and 6. The travel time t_{ij} between customer i and j , and corresponding cost c_{ij} is

$$c_{ij} = t_{ij} = |j - i| \quad i = 1, \dots, 6, \quad j = 0, \dots, 6$$

There are at most five vehicles available, and the cost of a vehicle is 6 units, i.e. $c_{0j} = 6$. The travel time from the depot to the first customer is $t_{0j} = 1$.

- 1 Find a feasible initial solution to the problem.
- 2 Use column generation to solve the problem to LP-optimality.
- 3 Write up all feasible routes, and solve the problem using CPLEX to integer optimality.
- 4 Compare the LP solution with the integer solution.

Graph



$$c_{ij} = t_{ij} = |j - i| \qquad c_{0j} = 6, \quad t_{0j} = 1$$

Possible tours (in lexicographic order, dominated removed)

tour	cost
0,1,2,4,6	11
0,(wait),2,4,6	10
0,3,2,4,6	11
0,3,(wait),4,6	10
0,3,5,6	9
0,3,5,4,6	11
0,(wait),(wait),4,6	8
0,(wait),5,4,6	10
0,(wait),5,6	7

Column Generation

- **Initial Solution.** We can choose the trivial routes:

tour	cost
0,1,6	11
0,2,6	10
0,3,6	9
0,4,6	8
0,5,6	7

which gives a solution of 45.

- **Column generation.** We start with formulation

$\begin{array}{ll} \min & 11x_1 + 10x_2 + 9x_3 + 8x_4 + 7x_5 \\ \text{s.t.} & x_1 \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \geq 1 \\ & \qquad \qquad x_2 \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \geq 1 \\ & \qquad \qquad \qquad x_3 \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \geq 1 \\ & \qquad \qquad \qquad \qquad x_4 \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \geq 1 \\ & \qquad \qquad \qquad \qquad \qquad x_5 \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \geq 1 \end{array}$
--

the dual variables corresponding to the constraints are

$$y_1 = 11, \quad y_2 = 10, \quad y_3 = 9, \quad y_4 = 8, \quad y_5 = 7$$

The reduced cost of a tour is now the “original” cost of the edges minus the dual variables corresponding to the customers visited. The most beneficial tour is (0, 1, 2, 4, 6) having the cost $11 - (11 + 10 + 8) = -18$. Since the reduced cost of the new column is

negative, we add it to the problem, getting:

$$\begin{array}{rcccccc}
 \min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 \\
 \text{s.t.} & x_1 & & & & & +x_6 \geq 1 \\
 & & x_2 & & & & +x_6 \geq 1 \\
 & & & x_3 & & & \geq 1 \\
 & & & & x_4 & & +x_6 \geq 1 \\
 & & & & & x_5 & \geq 1
 \end{array}$$

the dual variables are

$$y_1 = 0, \quad y_2 = 3, \quad y_3 = 9, \quad y_4 = 8, \quad y_5 = 7$$

The most beneficial tour is $(0, 3, 5, 4, 6)$ having the reduced cost $11 - (9 + 8 + 7) = -13$, which is added to the problem:

$$\begin{array}{rcccccc}
 \min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & +11x_7 \\
 \text{s.t.} & x_1 & & & & & +x_6 & \geq 1 \\
 & & x_2 & & & & +x_6 & \geq 1 \\
 & & & x_3 & & & +x_7 & \geq 1 \\
 & & & & x_4 & & +x_6 & +x_7 \geq 1 \\
 & & & & & x_5 & +x_7 & \geq 1
 \end{array}$$

the dual variables are

$$y_1 = 1, \quad y_2 = 10, \quad y_3 = 4, \quad y_4 = 0, \quad y_5 = 7$$

Adding the tour $(0, 3, 2, 4, 6)$ at reduced cost $11 - (10 + 4 + 0) = -3$ gives

$$\begin{array}{rcccccc}
 \min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & +11x_7 & +11x_8 \\
 \text{s.t.} & x_1 & & & & & +x_6 & & \geq 1 \\
 & & x_2 & & & & +x_6 & +x_8 & \geq 1 \\
 & & & x_3 & & & +x_7 & +x_8 & \geq 1 \\
 & & & & x_4 & & +x_6 & +x_7 & +x_8 \geq 1 \\
 & & & & & x_5 & +x_7 & & \geq 1
 \end{array}$$

with dual variables

$$y_1 = 4, \quad y_2 = 7, \quad y_3 = 4, \quad y_4 = 0, \quad y_5 = 7$$

The most beneficial tour is $(0, 3, 5, 6)$, having reduced cost $9 - (4 + 7) = -2$. Thus

$\min 11x_1 + 10x_2 + 9x_3 + 8x_4 + 7x_5 + 11x_6 + 11x_7 + 11x_8 + 9x_9$								
s.t.	x_1					$+x_6$		≥ 1
		x_2				$+x_6$	$+x_8$	≥ 1
			x_3				$+x_7$	$+x_8$
				x_4		$+x_6$	$+x_7$	$+x_8$
					x_5		$+x_7$	$+x_9$
								≥ 1

with dual variables

$$y_1 = 2, \quad y_2 = 9, \quad y_3 = 2, \quad y_4 = 0, \quad y_5 = 7$$

Now, all tours have reduced costs ≥ 0 , so we are done. The bound found by column generation is thus $z_{LP} = 20$.

- **Write up all feasible solutions.** The problem with all 9 tours added to the formulation, solved to IP-optimality, gives the solution $z_{IP} = 20$.
- **Compare bounds.** We have $z_{IP} = z_{LP}$ so the bounds are tight.