Friday, November 21

Program of the day:

- Example: Prize Collecting Traveling Salesman Problem
- Relaxation strategies.
- Lagrangian relaxation.

Cover inequalities

Consider the set

$$X = \left\{ x \in \mathbb{B}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}$$

We assume that $a_j \geq 0$ and $b \geq 0$.

Cover

A set $C \subseteq N$ is a cover if

$$\sum_{j \in C} a_j > b$$

A set $C \subseteq N$ is a minimal cover if $C \setminus \{j\}$ is not a cover for any $j \in C$

Cover Inequality

If C is a cover the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid for X.

Cover inequalities

- Order the variables so that $a_1 \geq a_2 \geq \ldots \geq a_n$
- Let $C = \{j_1, j_2, \ldots, j_n\}$ be a cover where $j_1 < j_2 < \ldots < j_n$
- Let $p = \min\{j : j \in N \setminus E(C)\}$
- The cover inequality

  $$\sum_{j \in E(C)} x_j \leq |C| - 1$$

  is a facet of $\text{conv}(X)$ if one of the following holds
  - $C = N$
  - $E(C) = N$ and $\sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_1 \leq b$
  - $C = E(C)$ and $\sum_{j \in C \setminus \{j_1\}} a_j + a_p \leq b$
  - $C \subset E(C) \subset N$ and $\sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_1 \leq b$ and $\sum_{j \in C \setminus \{j_1\}} a_j + a_p \leq b$

Example: Prize Collecting Traveling Salesman Problem

- Set of N cities.
- Salesman starts in city 1.
- To each edge e is associated a cost c_e
- To each node j is associated a profit f_j
- Visit *at least* two other cities
- Maximize profit − cost.

Introduce variables

- $x_e = 1$ if edge e is used.
- $y_j = 1$ if node j is visited.

Formulation

$$\begin{array}{lll}
\max & \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e & \\
\text{s.t.} & \sum_{e \in \delta(j)} x_e = 2 y_j & , \; j \in N \\
& \sum_{e \in E(S)} x_e \leq \sum_{i \in S \setminus \{k\}} y_i & , \; k \in S, S \subseteq N \setminus \{1\} \\
& y_1 = 1 & \\
& x_e \in \{0,1\}, y_j \in \{0,1\} &
\end{array}$$

Separation for generalized subtour constraints

Assume that we solve the ILP-problem

$$
\begin{array}{lll}
\max & \sum_{j\in N} f_j y_j - \sum_{e\in E} c_e x_e & \\
\text{s.t.} & \sum_{e\in\delta(j)} x_e = 2y_j & , \; j \in N \\
& y_1 = 1 & \\
& x \in \{0,1\}, y \in \{0,1\} &
\end{array} \tag{1}
$$

getting a solution (x^*, y^*). How do we find a violated GSE constraint?

- $N' = N \setminus 1$
- $E' = E \setminus \{\delta(1)\}$
- $z_i = 1$ iff $i \in S$

A constraint for (k, S) is violated if

$$\sum_{e\in E'(S)} x_e^* > \sum_{i\in S\setminus\{k\}} y_i^*$$

This can be formulated as a maximization problem

$$
\begin{array}{ll}
\gamma = \max & \sum_{e\in E'} x_e^* z_i z_j - \sum_{i\in N'\setminus\{k\}} y_i^* z_i \\
\text{s.t.} & z_k = 1 \\
& z \in \{0,1\}
\end{array}
$$

Separation for generalized subtour constraints

The quadratic 0-1 program

$$
\begin{array}{ll}
\gamma = \max & \sum_{e=(i,j)\in E'} x_e^* z_i z_j - \sum_{i\in N'\setminus\{k\}} y_i^* z_i \\
\text{s.t.} & z_k = 1 \\
& z \in \{0,1\}
\end{array}
$$

can be reformulated using

$$w_{(i,j)} = 1 \;\Leftrightarrow\; z_i = 1 \text{ and } z_j = 1$$

but since we maximize only

$$w_{(i,j)} = 1 \;\Rightarrow\; z_i = 1 \text{ and } z_j = 1$$

is needed

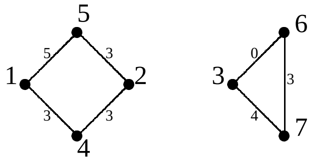
$$
\begin{array}{lll}
\gamma = \max & \sum_{e=(i,j)\in E'} x_e^* w_e - \sum_{i\in N'\setminus\{k\}} y_i^* z_i & \\
\text{s.t.} & w_{(i,j)} \le z_i & , \; (i,j) \in E' \\
& w_{(i,j)} \le z_j & , \; (i,j) \in E' \\
& z_k = 1 & \\
& w \in \{0,1\}, z \in \{0,1\} &
\end{array}
$$

This formulation is TU and thus can be solved in polynomial time

Separation for generalized subtour constraints

$$
f = (2,4,1,3,7,1,7) \text{ and}
$$

$$
c_e = \begin{pmatrix}
- & 4 & 3 & 3 & 5 & 2 & 5 \\
- & - & 5 & 3 & 3 & 4 & 7 \\
- & - & - & 4 & 6 & 0 & 4 \\
- & - & - & - & 4 & 4 & 6 \\
- & - & - & - & - & 5 & 8 \\
- & - & - & - & - & - & 3 \\
- & - & - & - & - & - & -
\end{pmatrix}
$$



The LP-relaxation of (1) gives the routes

$$(1,5,2,4) \text{ and } (3,6,7)$$

The separation algorithm returns

$$x_{36} + x_{37} + x_{67} \le y_3 + y_7$$

which cutts off the subtour $(3,6,7)$.

Relaxation

In a branch-and-bound algorithm we find upper bounds by relaxing the problem

Relaxation (Wolsey sec. 2.1)

$$
\begin{array}{ll}
\max\{cx : x \in S\} & (IP) \\
\max\{f(x) : x \in T\} & (RP)
\end{array}
$$

RP is a relaxation of IP if

- $S \subseteq T$
- $f(x) \ge cx$ for all $x \in S$

Which constraints should be relaxed?

- Quality of bound (tightness of relaxation)
- Remaining problem can be solved efficiently
- Proper multipliers can be found efficiently
- Constraints difficult to formulate mathematically
- Constraints which are too expensive to write up

Overview

Different relaxations

- LP-relaxation
- Deleting constraint
- Lagrangian relaxation
- Surrogate relaxation
- Semidefinite relaxation

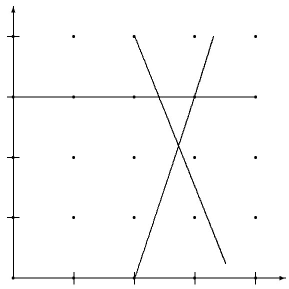
Relaxations are often used in combination.

Hierarchy

- Best surrogate relaxation
- Best lagrangian relaxation
- LP-relaxation

Lagrangian relaxation, example

$$\begin{array}{llcll} \text{maximize} & 4x_1 & + & x_2 & \\ \text{subject to} & 3x_1 & - & x_2 & \leq 6 \\ & & & x_2 & \leq 3 \\ & 5x_1 & + & 2x_2 & \leq 18 \\ & x_1, & & x_2 & \geq 0, \text{ integer} \end{array}$$



IP solution $(x_1, x_2) = (2, 3)$ with $z_{IP} = 11$
LP solution $(x_1, x_2) = (\frac{30}{11}, \frac{24}{11})$ with $z_{LP} = \frac{144}{11} = 13.1$

Last constraint complicating, relax using multiplier $\lambda = \frac{1}{2}$

$$\begin{array}{llcll} \text{maximize} & 4x_1 & + & x_2 & - \frac{1}{2}(5x_1 + 2x_2 - 18) = \frac{3}{2}x_1 + 9 \\ \text{subject to} & 3x_1 & - & x_2 & \leq 6 \\ & & & x_2 & \leq 3 \\ & x_1, & & x_2 & \geq 0, \text{ integer} \end{array}$$

Solution $(x_1, x_2) = (3, 3)$ with $z_{LR} = \frac{3}{2}3 + 9 = 13.5$
Upper bound

Lagrangian relaxation

Integer Programming Problem

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & Dx \leq d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Lagrange relax $Dx \leq d$, using multipliers $\lambda \geq 0$

$$\begin{array}{ll} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \leq b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Proposition 1 Optimal solution to relaxed problem gives upper bound on original problem

Proof show that relaxation

multiplier λ_i "punishment"
If λ_i large $\Rightarrow$ constraint satisfied
If $\lambda_i = 0 \Rightarrow$ drop constrain

Lagrangian relaxation

Lagrange relaxed problem as function of $\lambda \geq 0$

$$\begin{array}{ll} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \leq b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

Natural questions:

- How do we find best λ?
- How tight is relaxation?

Properties of Lagrange relaxation

Geom. interpretation, Lagrangian Relaxation

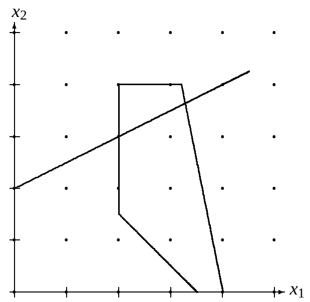
$$
\begin{array}{llcrcr}
\max & 7x_1 & + & 2x_2 & & \\
\text{s.t.} & -x_1 & + & 2x_2 & \leq & 4 \\
 & 5x_1 & + & x_2 & \leq & 20 \\
 & -2x_1 & - & 2x_2 & \leq & -7 \\
 & -x_1 & & & \leq & -2 \\
 & & & x_2 & \leq & 4 \\
 & & & \multicolumn{3}{l}{x_1, x_2 \text{ integer}}
\end{array}
$$

First constraint "$-x_1 + 2x_2 \leq 4$" is "complicating"
Lagrangian relax this constraint ($\lambda \geq 0$) getting objective

$$7x_1 + 2x_2 - \lambda(-x_1 + 2x_2 - 4)$$

Relaxed problem

$$
\begin{array}{llcrcr}
\max & \multicolumn{5}{l}{(7+\lambda)x_1 + (2-2\lambda)x_2 + 4\lambda} \\
\text{s.t.} & 5x_1 & + & x_2 & \leq & 20 \\
 & -2x_1 & - & 2x_2 & \leq & -7 \\
 & -x_1 & & & \leq & -2 \\
 & & & x_2 & \leq & 4 \\
 & & & \multicolumn{3}{l}{x_1, x_2 \text{ integer}}
\end{array}
$$



Geom. interpretation, Lagrangian Relaxation

Original problem, integer solution

$(x_1, x_2) = (4, 0)$ $\qquad z = 28.00$

Original problem, LP-relaxed solution

$$(x_1, x_2) = (\frac{36}{11}, \frac{40}{11}) = (3.27, 3.64) \qquad z = 30.18$$

Drop first constraint, integer solution

$(x_1, x_2) = (3, 4)$ $\qquad z = 29.00$

Drop first constraint, LP-relaxed solution

$$(x_1, x_2) = (\frac{16}{5}, 4) = (3.2, 4) \qquad z = 30.40$$

Maximum on Q, LP-relaxed solution

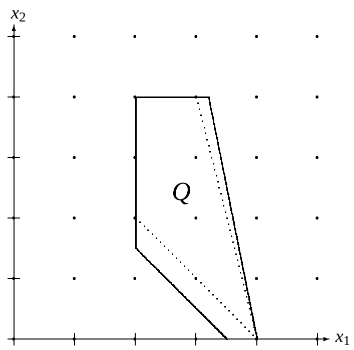
$(x_1, x_2) = (3, 4)$ $\qquad z = 29.00$

Maximum on Q, with first constraint added

$$(x_1, x_2) = (\frac{28}{9}, \frac{32}{9}) = (3.11, 3.56) \qquad z = 28.88$$

Geom. interpretation, Lagrangian Relaxation

Viewpoint 1: fixed λ



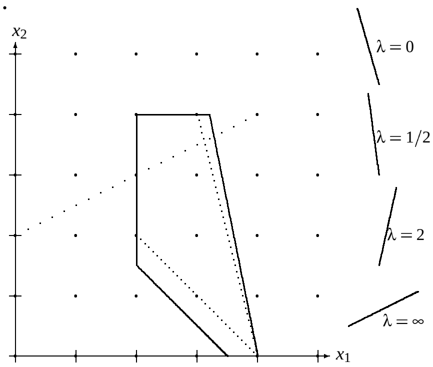
$$
\begin{array}{llcrcr}
\max & \multicolumn{5}{l}{(7+\lambda)x_1 + (2-2\lambda)x_2 + 4\lambda} \\
\text{s.t.} & 5x_1 & + & x_2 & \leq & 20 \\
 & -2x_1 & - & 2x_2 & \leq & -7 \\
 & -x_1 & & & \leq & -2 \\
 & & & x_2 & \leq & 4 \\
 & & & \multicolumn{3}{l}{x_1, x_2 \text{ integer}}
\end{array}
$$

Redefinition using convex hull of Q

$$
\max \quad (7+\lambda)x_1 + (2-2\lambda)x_2 + 4\lambda
$$
$$
\text{s.t.} \quad \left.
\begin{array}{rcrcr}
4x_1 & + & x_2 & \leq & 16 \\
-x_1 & - & x_2 & \leq & -4 \\
-x_1 & & & \leq & -2 \\
 & & x_2 & \leq & 4
\end{array}
\right\} \quad Q
$$

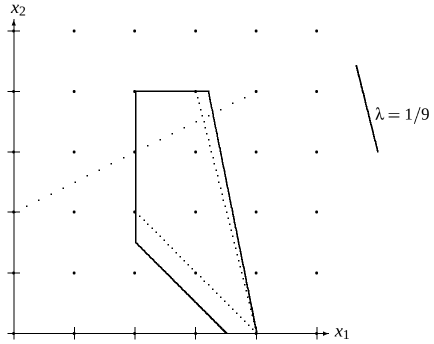
Geom. interpretation, Lagrangian Relaxation

Viewpoint 1:



- λ is a modifier of the objective function
- For $0 \leq \lambda \leq \frac{1}{9}$, optimal solution $(3, 4)$
  $$z_{LR}(\lambda) = (7+\lambda)3 + (2-2\lambda)4 + 4\lambda = 29 - \lambda$$
- For $\lambda \geq \frac{1}{9}$ optimal solution $(4, 0)$
  $$z_{LR}(\lambda) = (7+\lambda)4 + (2-2\lambda)0 + 4\lambda = 28 + 8\lambda$$
- Increasing lambda is forcing the optimal solution to satisfy relaxed constraint.

Geom. interpretation, Lagrangian Relaxation



- When $\lambda = \frac{1}{9}$ we get the tightest bound.
- In this case the isoprofit line is parallel to the line through $(3,4)$ and $(4,0)$.
- We may choose an arbitrary point x^* on this line
$$(x_1^*, x_2^*) = (\frac{28}{9}, \frac{32}{9}) = (3.11, 3.56)$$
which satisfies the relaxed constraint
$$-x_1 + 2x_2 \leq 4$$
- In this case
$$z_{LD} = \max\{cx : Dx \leq d, x \in \text{conv}(Q)\}$$

This "proves" theorem 10.3 page 172.

Geom. interpretation, Lagrangian Relaxation

Integer Programming Problem

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & Dx \leq d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

$$\boxed{\max\left\{cx : x \in \text{conv}(Ax \leq b, Dx \leq d, x \in \mathbb{Z}_+)\right\}}$$

Lagrange Relaxation, multipliers $\lambda \geq 0$

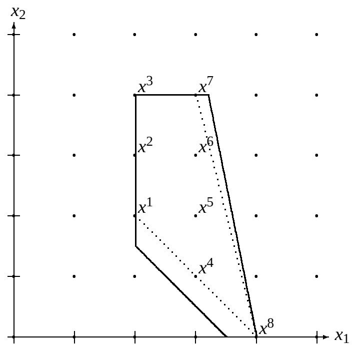
$$\begin{array}{ll} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \leq b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

for best multiplier $\lambda \geq 0$

$$\boxed{\max\left\{cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+)\right\}}$$

Geom. interpretation, Lagrangian Relaxation

Viewpoint 2: fixed point x^i



There are 8 integer points in Q:

$$\{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8\} = \\ \{(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,0)\}$$

For fixed x^i the objective function

$$z_{LR}(\lambda, x^i) = (7+\lambda)x_1^i + (2-2\lambda)x_2^i + 4\lambda = 7x_1^i + 2x_2^i + \lambda(x_1^i - 2x_2^i + 4)$$

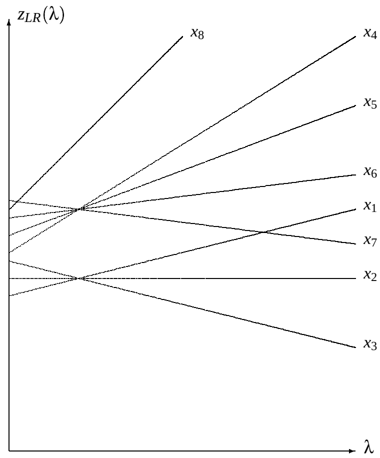
is an affine function.

E.g. for $x^7 = (3,4)$

$$z_{LR}(\lambda, x^7) = 7 \cdot 3 + 2 \cdot 4 + \lambda(3 - 2 \cdot 4 + 4) = 29 - \lambda$$

Geom. interpretation, Lagrangian Relaxation

Viewpoint 2:



Objective

$$z_{LR}(\lambda) = \max_{x^i \in Q} z(\lambda, x^i)$$

Proposition 2 The lagrangian relaxed problem $z_{LR}(\lambda)$ as function of the multipliers $\lambda \in \mathbb{R}$, $\lambda \geq 0$ is picewise linear and convex

(see Wolsey, figure page 173)

Lagrangian relaxation and duality

- Lagrangian relaxation is a generalization of duality, where we may relax any subset of constraints.

- Lagrange Relaxation

$$\begin{array}{ll} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \le b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{array}$$

  Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \ge 0} z_{LR}(\lambda)$$

  is an LP-problem

- Optimal multipliers λ may be found by simplex.

- Subgradient is however faster when few iterations.

Lagrangian Relaxation

Integer Programming Problem

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \le b \\ & Dx \le d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{array}$$

Lagrange Relaxation, multipliers $\lambda \ge 0$

$$\begin{array}{ll} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \le b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{array}$$

Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \ge 0} z_{LR}(\lambda)$$

Assume that the "nice constraints" $Ax \le b$ define the convex hull, e.g.

- A is totally unimodular, and b is a vector of integers
- There are no constraints left
- The remaining constraints are defined in linear variables

Lagrangian Relaxation

for best multiplier $\lambda \ge 0$ strength of model

$$\boxed{\max \left\{ cx : Dx \le d, x \in \text{conv}(Ax \le b, x \in \mathbb{Z}_+) \right\}}$$

If $\{x : Ax \le b\} = \{x \in \text{conv}(Ax \le b, x \in \mathbb{Z}_+)\}$ strength

$$\boxed{\max \left\{ cx : Dx \le d, Ax \le b \right\}}$$

Corrollary (page 173 in Wolsey)

$$z_{LD} = z_{LP}$$

for any objective function cx.

- We do not obtain better bounds than by linear relaxation.
- We may find $z_{LP} = z_{LD}$ in polynomial time.
- If the remaining problem $Ax \le b$ has a nice structure (e.g. min-spanning-tree) we may find z_{LD} faster than z_{LP}.

Lagrangian Relaxation

How should we choose the optimal multipliers λ (i.e. the multipliers which minimize z_{LD}) in this case

Consider z_{LP} as solution to

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \le b \\ & A'x \le b' \\ & x \ge 0 \end{array}$$

where the dual problem is

$$\begin{array}{ll} \text{minimize} & by + b'y' \\ \text{subject to} & yA + y'A' \ge c \\ & y, y' \ge 0 \end{array}$$

Now consider the lagrangian relaxed problem $z_{LR}(\lambda)$

$$\begin{array}{ll} \lambda b' + \text{maximize} & (c - \lambda A')x \\ \text{subject to} & Ax \le b \\ & x \ge 0 \quad \text{(IP sol. for free)} \end{array}$$

where the dual problem is

$$\begin{array}{ll} \lambda b' + \text{minimize} & by \\ \text{subject to} & yA \ge c - \lambda A' \\ & \lambda, y \ge 0 \end{array}$$

Lagrangian Relaxation

Thus

- If relax all constraints: ordinary dual problem
- Lagrangian relaxation of a constraint can be seen as "dualization" of a constraint.
- We have found a technique for deriving the best lagrangian multipliers in some special cases.