Friday, November 21

Program of the day:
- Example: Prize Collecting Traveling Salesman Problem
- Relaxation strategies.
- Lagrangian relaxation.

Cover inequalities
Consider the set
\[ X = \left\{ x \in \mathbb{B}^n : \sum_{j=1}^n a_j x_j \leq b \right\} \]

We assume that \( a_j \geq 0 \) and \( b \geq 0 \).

Cover
A set \( C \subseteq N \) is a cover if
\[ \sum_{j \in C} a_j > b \]

A set \( C \subseteq N \) is a minimal cover if \( C \setminus \{j\} \) is not a cover for any \( j \in C \).

Cover Inequality
If \( C \) is a cover the cover inequality
\[ \sum_{j \in C} x_j \leq |C| - 1 \]
is valid for \( X \).

Example: Prize Collecting Traveling Salesman Problem
- Set of \( N \) cities.
- Salesman starts in city 1.
- To each edge \( e \) is associated a cost \( c_e \)
- To each node \( j \) is associated a profit \( f_j \)
- Visit at least two other cities
- Maximize profit – cost.

Introduce variables
- \( x_e = 1 \) if edge \( e \) is used.
- \( y_j = 1 \) if node \( j \) is visited.

Formulation
\[
\begin{align*}
\max & \quad \sum_{j \in S} f_j y_j - \sum_{e \in k} c_e x_e \\
\text{s.t.} & \quad \sum_{e \in S} x_e = 2y_j, \quad j \in N \\
& \quad \sum_{e \in S} x_e - \sum_{i \in S \setminus \{k\}} y_i \leq 0, \quad k \in S, S \subseteq N \setminus \{1\} \\
& \quad y_1 = 1 \\
& \quad x_e \in \{0,1\}, y_j \in \{0,1\}
\end{align*}
\]
Separation for generalized subtour constraints

Assume that we solve the ILP-problem
\[
\begin{align*}
\max & \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e \\
\text{s.t.} & \sum_{e \in \delta(j)} x_e = 2y_j, \quad j \in N \\
& y_1 = 1 \\
& x \in \{0,1\}, y \in \{0,1\}
\end{align*}
\]
getting a solution \((x^*, y^*)\). How do we find a violated GSE constraint?

- \(N' = N \setminus 1\)
- \(E' = E \setminus \delta(1)\)
- \(z_i = 1\) iff \(i \in S\)

A constraint for \((k, S)\) is violated if
\[
\sum_{e \in E(S)} x^*_e > \sum_{i \in N(k)} y^*_i
\]
This can be formulated as a maximization problem
\[
\gamma = \max \sum_{e \in E'(S)} x^*_e z_e - \sum_{i \in N(k)} y^*_i z_i \\
\text{s.t.} \quad z_k = 1 \\
\quad z \in \{0,1\}
\]

The LP-relaxation of (1) gives the routes
\((1,5,2,4)\) and \((3,6,7)\)
The separation algorithm returns
\[x_{36} + x_{37} + x_{67} \leq y_3 + y_7\]
which cuts off the subtour \((3,6,7)\).

Separation for generalized subtour constraints

The quadratic 0-1 program
\[
\begin{align*}
\gamma = \max & \sum_{e \in E'} x^*_e z_e - \sum_{i \in N(k)} y^*_i z_i \\
\text{s.t.} & \quad z_k = 1 \\
& \quad z \in \{0,1\}
\end{align*}
\]
can be reformulated using
\[
w_{(i,j)} = 1 \iff z_i = 1 \quad z_j = 1
\]
but since we maximize only
\[
w_{(i,j)} = 1 \Rightarrow z_i = 1 \quad z_j = 1
\]
is needed
\[
\gamma = \max \sum_{e \in E'} x^*_e w_e - \sum_{i \in N(k)} y^*_i z_i \\
\text{s.t.} \quad w_{(i,j)} \leq z_i \quad (i,j) \in E' \\
\quad w_{(i,j)} \leq z_j \quad (i,j) \in E' \\
\quad z_k = 1 \\
\quad w \in \{0,1\}, z \in \{0,1\}
\]
This formulation is TU and thus can be solved in polynomial time.

Relaxation

In a branch-and-bound algorithm we find upper bounds by relaxing the problem

Relaxation (Wolsey sec. 2.1)
\[
\begin{align*}
\max \{cx : x \in S\} & \quad (IP) \\
\max \{f(x) : x \in T\} & \quad (RP)
\end{align*}
\]
RP is a relaxation of IP if
- \(S \subseteq T\)
- \(f(x) \geq cx\) for all \(x \in S\)

Which constraints should be relaxed?
- Quality of bound (tightness of relaxation)
- Remaining problem can be solved efficiently
- Proper multipliers can be found efficiently
- Constraints difficult to formulate mathematically
- Constraints which are too expensive to write up
Overview

Different relaxations
- LP-relaxation
- Deleting constraint
- Lagrangian relaxation
- Surrogate relaxation
- Semidefinite relaxation

Relaxations are often used in combination.

Hierarchy
- Best surrogate relaxation
- Best lagrangian relaxation
- LP-relaxation

Lagrangian relaxation, example

maximize $4x_1 + x_2$
subject to $3x_1 - x_2 \leq 6$
$x_2 \leq 3$
$5x_1 + 2x_2 \leq 18$
$x_1, x_2 \geq 0$, integer

IP solution $(x_1, x_2) = (2, 3)$ with $z_{IP} = 11$
LP solution $(x_1, x_2) = \left(\frac{30}{11}, \frac{24}{11}\right)$ with $z_{LP} = \frac{144}{11} = 13.1$

Last constraint complicating, relax using multiplier $\lambda = \frac{1}{2}$

maximize $4x_1 + x_2 - \frac{1}{2}(5x_1 + 2x_2 - 18) - \frac{1}{2}x_1 + 9$
subject to $3x_1 - x_2 \leq 6$
$x_2 \leq 3$
$x_1, x_2 \geq 0$, integer

Solution $(x_1, x_2) = (3, 3)$ with $z_{LR} = \frac{1}{2}3 + 9 = 13.5$
Upper bound

Lagrangian relaxation

Integer Programming Problem

maximize $cx$
subject to $Ax \leq b$
$Dx \leq d$
$x_j \in \mathbb{Z}_+,$ $j = 1, \ldots, n$

Lagrange relax $Dx \leq d$, using multipliers $\lambda \geq 0$

maximize $z_{LR}(\lambda) = cx - \lambda(Dx - d)$
subject to $Ax \leq b$
$x_j \in \mathbb{Z}_+,$ $j = 1, \ldots, n$

Proposition 1 Optimal solution to relaxed problem gives upper bound on original problem

Proof show that relaxation

multiplier $\lambda_i$ “punishment”
If $\lambda_i$ large $\Rightarrow$ constraint satisfied
If $\lambda_i = 0 \Rightarrow$ drop constrain

Lagrangian relaxation

Lagrange relaxed problem as function of $\lambda \geq 0$

maximize $z_{LR}(\lambda) = cx - \lambda(Dx - d)$
subject to $Ax \leq b$
$x_j \in \mathbb{Z}_+,$ $j = 1, \ldots, n$

Lagrangian Dual Problem

$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$

Natural questions:
- How do we find best $\lambda$?
- How tight is relaxation?

Properties of Lagrange relaxation
Geom. interpretation, Lagrangian Relaxation

\[
\text{max} \quad 7x_1 + 2x_2 \\
\text{s.t.} \quad \begin{align*}
5x_1 + 2x_2 & \leq 4 \\
-2x_1 - 2x_2 & \leq 20 \\
-x_1 - 2x_2 & \leq -2 \\
x_1, x_2 & \geq 4 \\
x_1, x_2 & \geq 0 
\end{align*}
\]

Drop first constraint, getting objective.

Increasing lambda is forcing the optimal solution to satisfy relaxed constraint.

Viewpoint 1: fixed \( \lambda \)

Geom. interpretation, Lagrangian Relaxation

\[
\text{max} \quad (7 + \lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda \\
\text{s.t.} \quad \begin{align*}
5x_1 + x_2 & \leq 20 \\
-2x_1 - 2x_2 & \leq -7 \\
-x_1 - 2x_2 & \leq -2 \\
x_1, x_2 & \leq 4 \\
x_1, x_2 & \geq 0 
\end{align*}
\]

Redefinition using convex hull of \( Q \)

Geom. interpretation, Lagrangian Relaxation

Original problem, integer solution

\( (x_1, x_2) = (4, 0) \quad z = 28.00 \)

Original problem, LP-relaxed solution

\( (x_1, x_2) = \left( \frac{36}{11}, \frac{40}{11} \right) = (3.27, 3.64) \quad z = 30.18 \)

Drop first constraint, integer solution

\( (x_1, x_2) = (3, 4) \quad z = 29.00 \)

Drop first constraint, LP-relaxed solution

\( (x_1, x_2) = \left( \frac{16}{5}, 4 \right) = (3.2, 4) \quad z = 30.40 \)

Maximum on \( Q \), LP-relaxed solution

\( (x_1, x_2) = (3, 4) \quad z = 29.00 \)

Maximum on \( Q \), with first constraint added

\( (x_1, x_2) = \left( \frac{28}{9}, \frac{32}{9} \right) = (3.11, 3.56) \quad z = 28.88 \)

Viewpoint 1:
When $\lambda = \frac{1}{9}$ we get the tightest bound.

In this case the isoprofit line is parallel to the line through $(3,4)$ and $(4,0)$.

We may choose an arbitrary point $x^*$ on this line

$$\left(x_1^*, x_2^*\right) = \left(\frac{28}{9}, \frac{32}{9}\right) = (3.11, 3.56)$$

which satisfies the relaxed constraint

$$-x_1 + 2x_2 \leq 4$$

In this case

$$z_{LR} = \max \{ cx : Dx \leq d, x \in \text{conv}(Q) \}$$

This “proves” theorem 10.3 page 172.

There are 8 integer points in $Q$:

$$\{x_1, x_2, x_3, x_4, x_5, x_7, x_8\} = \{(2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,0)\}$$

For fixed $x^t$ the objective function

$$z_{LR}(\lambda, x^t) = (7 + \lambda)x_1^t + (2 - 2\lambda)x_2^t + 4\lambda - 7x_1^t + 2x_2^t + \lambda(x_1^t - 2x_2^t + 4)$$

is an affine function.

E.g. for $x^t = (3, 4)$

$$z_{LR}(\lambda, x^t) = 7 \cdot 3 + 2 \cdot 4 + \lambda(3 - 2 \cdot 4 + 4) = 29 - \lambda$$

The lagrangian relaxed problem $z_{LR}(\lambda)$ as function of the multipliers $\lambda \in \mathbb{R}$, $\lambda \geq 0$ is piecewise linear and convex.

(see Wolsey, figure page 173)
Lagrangian relaxation and duality

- Lagrangian relaxation is a generalization of duality, where we may relax any subset of constraints.

- Lagrange Relaxation
  \[
  \text{maximize } z_{LR}(\lambda) = cx - \lambda(Dx - d)
  \]
  subject to \( Ax \leq b \)
  \[ x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \]

Lagrangian Dual Problem
  \[
  z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)
  \]
  is an LP-problem

- Optimal multipliers \( \lambda \) may be found by simplex.

- Subgradient is however faster when few iterations.

Lagrangian Relaxation

for best multiplier \( \lambda \geq 0 \) strength of model

\[
\max \left\{ cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+) \right\}
\]

If \( \{x : Ax \leq b\} = \{x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+)\} \) strength

\[
\max \left\{ cx : Dx \leq d, Ax \leq b \right\}
\]

Corollary (page 173 in Wolsey)

\[
z_{LD} = z_{LP}
\]

for any objective function \( cx \).

- We do not obtain better bounds than by linear relaxation.

- We may find \( z_{LP} = z_{LD} \) in polynomial time.

- If the remaining problem \( Ax \leq b \) has a nice structure (e.g. min-spanning-tree) we may find \( z_{LD} \) faster than \( z_{LP} \).

Lagrangian Relaxation

Integer Programming Problem

\[
\begin{align*}
\text{maximize } & cx \\
\text{subject to } & Ax \leq b \\
& Dx \leq d \\
x_j & \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

Lagrange Relaxation, multipliers \( \lambda \geq 0 \)

\[
\begin{align*}
\text{maximize } & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\
\text{subject to } & Ax \leq b \\
x_j & \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

Lagrangian Dual Problem

\[
z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)
\]

Assume that the “nice constraints” \( Ax \leq b \) define the convex hull, e.g.

- \( A \) is totally unimodular, and \( b \) is a vector of integers

- There are no constraints left

- The remaining constraints are defined in linear variables

Lagrangian Relaxation

How should we choose the optimal multipliers \( \lambda \) (i.e. the multipliers which minimize \( z_{LD} \)) in this case

Consider \( z_{LP} \) as solution to

\[
\begin{align*}
\text{maximize } & cx \\
\text{subject to } & Ax \leq b \\
& A'x \leq b' \\
x & \geq 0
\end{align*}
\]

where the dual problem is

\[
\begin{align*}
\text{minimize } & by + b'y' \\
\text{subject to } & yA + y'A' \geq c \\
y, y' & \geq 0
\end{align*}
\]

Now consider the lagrangian relaxed problem \( z_{LR}(\lambda) \)

\[
\lambda d' + \max \left\{ (c - \lambda A')x : Ax \leq b \\
x \geq 0 \right\} \quad (\text{IP sol. for free})
\]

where the dual problem is

\[
\begin{align*}
\lambda d' + \min \left\{ by : A'y + y'A' \geq c - \lambda A' \\
\lambda, y & \geq 0
\end{align*}
\]
Lagrangian Relaxation

Thus
- If relax all constraints: ordinary dual problem
- Lagrangian relaxation of a constraint can be seen as “dualization” of a constraint.
- We have found a technique for deriving the best lagrangian multipliers in some special cases.