Friday, November 21

Program of the day:

- Example: Prize Collecting Traveling Salesman Problem
- Relaxation strategies.
- Lagrangian relaxation.
Cover inequalities

Consider the set

\[
X = \left\{ x \in \mathbb{B}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}
\]

We assume that \( a_j \geq 0 \) and \( b \geq 0 \).

Cover

A set \( C \subseteq N \) is a cover if

\[
\sum_{j \in C} a_j > b
\]

A set \( C \subseteq N \) is a minimal cover if \( C \setminus \{j\} \) is not a cover for any \( j \in C \).

Cover Inequality

If \( C \) is a cover the cover inequality

\[
\sum_{j \in C} x_j \leq |C| - 1
\]

is valid for \( X \).
Cover inequalities

- Order the variables so that $a_1 \geq a_2 \geq \ldots \geq a_n$
- Let $C = \{j_1, j_2, \ldots, j_n\}$ be a cover where $j_1 < j_2 < \ldots < j_n$
- Let $p = \min\{j : j \in N \setminus E(C)\}$
- The cover inequality

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is a facet of $\text{conv}(X)$ if one of the following holds

- $C = N$
- $E(C) = N$ and $\sum_{j \in C\setminus\{j_1, j_2\}} a_j + a_1 \leq b$
- $C = E(C)$ and $\sum_{j \in C\setminus\{j_1\}} a_j + a_p \leq b$
- $C \subset E(C) \subset N$ and $\sum_{j \in C\setminus\{j_1, j_2\}} a_j + a_1 \leq b$ and $\sum_{j \in C\setminus\{j_1\}} a_j + a_p \leq b$
Example: Prize Collecting Traveling Salesman Problem

- Set of $N$ cities.
- Salesman starts in city 1.
- To each edge $e$ is associated a cost $c_e$
- To each node $j$ is associated a profit $f_j$
- Visit at least two other cities
- Maximize profit – cost.

Introduce variables

- $x_e = 1$ if edge $e$ is used.
- $y_j = 1$ if node $j$ is visited.

Formulation

$$\max \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e$$

s.t. \[ \sum_{e \in \delta(j)} x_e = 2y_j, \quad j \in N \]

\[ \sum_{e \in E(S)} x_e \leq \sum_{i \in S \setminus \{k\}} y_i, \quad k \in S, S \subseteq N \setminus \{1\} \]

$y_1 = 1$

$x_e \in \{0, 1\}, y_j \in \{0, 1\}$
Separation for generalized subtour constraints

Assume that we solve the ILP-problem

\[
\begin{align*}
\max & \quad \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(j)} x_e = 2y_j, \quad j \in N \\
& \quad y_1 = 1 \\
& \quad x \in \{0, 1\}, y \in \{0, 1\}
\end{align*}
\]

getting a solution \((x^*, y^*)\). How do we find a violated GSE constraint?

- \(N' = N \setminus 1\)
- \(E' = E \setminus \{\delta(1)\}\)
- \(z_i = 1\) iff \(i \in S\)

A constraint for \((k, S)\) is violated if

\[
\sum_{e \in E'(S)} x_e^* > \sum_{i \in S \setminus \{k\}} y_i^*
\]

This can be formulated as a maximization problem

\[
\gamma = \max \sum_{e \in E'} x_e^* z_{i.e} z_{j.e} - \sum_{i \in N' \setminus \{k\}} y_i^* z_i \\
\text{s.t.} \quad z_k = 1 \\
\quad z \in \{0, 1\}
\]
Separation for generalized subtour constraints

The quadratic 0-1 program

\[
\gamma = \max \sum_{e=(i,j) \in E'} x_e^* z_i z_j - \sum_{i \in N' \setminus \{k\}} y_i^* z_i
\]

s.t. \( z_k = 1 \)

\( z \in \{0, 1\} \)

can be reformulated using

\( w(i,j) = 1 \iff z_i = 1 \) and \( z_j = 1 \)

but since we maximize only

\( w(i,j) = 1 \Rightarrow z_i = 1 \) and \( z_j = 1 \)

is needed

\[
\gamma = \max \sum_{e=(i,j) \in E'} x_e^* w_e - \sum_{i \in N' \setminus \{k\}} y_i^* z_i
\]

s.t. \( w(i,j) \leq z_i \), \( (i, j) \in E' \)

\( w(i,j) \leq z_j \), \( (i, j) \in E' \)

\( z_k = 1 \)

\( w \in \{0, 1\} \), \( z \in \{0, 1\} \)

This formulation is TU and thus can be solved in polynomial time
Separation for generalized subtour constraints

\[ f = (2, 4, 1, 3, 7, 1, 7) \text{ and } \]

\[ c_e = \begin{pmatrix}
-4 & 3 & 3 & 5 & 2 & 5 \\
-5 & 3 & 3 & 4 & 7 \\
-4 & 6 & 0 & 4 \\
-4 & 6 & 4 & 8 \\
-5 & 8 \\
-3
\end{pmatrix} \]

The LP-relaxation of (1) gives the routes

\((1, 5, 2, 4)\) and \((3, 6, 7)\)

The separation algorithm returns

\[ x_{36} + x_{37} + x_{67} \leq y_3 + y_7 \]

which cutts off the subtour \((3, 6, 7)\).
Relaxation

In a branch-and-bound algorithm we find upper bounds by relaxing the problem

Relaxation (Wolsey sec. 2.1)

\[
\begin{align*}
\max \{ cx : x \in S \} & \quad (IP) \\
\max \{ f(x) : x \in T \} & \quad (RP)
\end{align*}
\]

RP is a relaxation of IP if

- \( S \subseteq T \)
- \( f(x) \geq cx \) for all \( x \in S \)

Which constraints should be relaxed?

- Quality of bound (tightness of relaxation)
- Remaining problem can be solved efficiently
- Proper multipliers can be found efficiently
- Constraints difficult to formulate mathematically
- Constraints which are too expensive to write up
Overview

Different relaxations

- LP-relaxation
- Deleting constraint
- Lagrangian relaxation
- Surrogate relaxation
- Semidefinite relaxation

Relaxations are often used in combination.

Hierarchy

- Best surrogate relaxation
- Best lagrangian relaxation
- LP-relaxation
Lagrangian relaxation, example

\[
\begin{align*}
\text{maximize} & \quad 4x_1 + x_2 \\
\text{subject to} & \quad 3x_1 - x_2 \leq 6 \\
& \quad x_2 \leq 3 \\
& \quad 5x_1 + 2x_2 \leq 18 \\
& \quad x_1, \quad x_2 \geq 0, \text{ integer}
\end{align*}
\]

IP solution \((x_1, x_2) = (2, 3)\) with \(z_{IP} = 11\)
LP solution \((x_1, x_2) = \left(\frac{30}{11}, \frac{24}{11}\right)\) with \(z_{LP} = \frac{144}{11} = 13.1\)

Last constraint complicating, relax using multiplier \(\lambda = \frac{1}{2}\)

\[
\begin{align*}
\text{maximize} & \quad 4x_1 + x_2 - \frac{1}{2}(5x_1 + 2x_2 - 18) = \frac{3}{2}x_1 + 9 \\
\text{subject to} & \quad 3x_1 - x_2 \leq 6 \\
& \quad x_2 \leq 3 \\
& \quad x_1, \quad x_2 \geq 0, \text{ integer}
\end{align*}
\]

Solution \((x_1, x_2) = (3, 3)\) with \(z_{LR} = \frac{3}{2}3 + 9 = 13.5\)
Upper bound
Lagrangian relaxation

Integer Programming Problem

maximize \( cx \)
subject to \( Ax \leq b \)
\( Dx \leq d \)
\( x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \)

Lagrange relax \( Dx \leq d \), using multipliers \( \lambda \geq 0 \)

maximize \( z_{LR}(\lambda) = cx - \lambda(Dx - d) \)
subject to \( Ax \leq b \)
\( x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \)

**Proposition 1** Optimal solution to relaxed problem gives upper bound on original problem

**Proof** show that relaxation

multiplier \( \lambda_i \) “punishment”
If \( \lambda_i \) large \( \Rightarrow \) constraint satisfied
If \( \lambda_i = 0 \) \( \Rightarrow \) drop constrain
Lagrangian relaxation

Lagrange relaxed problem as function of $\lambda \geq 0$

$$\text{maximize } z_{LR}(\lambda) = cx - \lambda(Dx - d)$$
subject to $Ax \leq b$
$$x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n$$

Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

Natural questions:

- How do we find best $\lambda$?
- How tight is relaxation?

Properties of Lagrange relaxation
Geom. interpretation, Lagrangian Relaxation

\[
\begin{align*}
\text{max} & \quad 7x_1 + 2x_2 \\
\text{s.t.} & \quad -x_1 + 2x_2 \leq 4 \\
& \quad 5x_1 + x_2 \leq 20 \\
& \quad -2x_1 - 2x_2 \leq -7 \\
& \quad -x_1 \leq -2 \\
& \quad x_2 \leq 4 \\
& \quad x_1, x_2 \text{ integer}
\end{align*}
\]

First constraint ”\(-x_1 + 2x_2 \leq 4\)” is “complicating”

Lagrangian relax this constraint (\(\lambda \geq 0\)) getting objective

\[
7x_1 + 2x_2 - \lambda(-x_1 + 2x_2 - 4)
\]

Relaxed problem

\[
\begin{align*}
\text{max} & \quad (7 + \lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda \\
\text{s.t.} & \quad 5x_1 + x_2 \leq 20 \\
& \quad -2x_1 - 2x_2 \leq -7 \\
& \quad -x_1 \leq -2 \\
& \quad x_2 \leq 4 \\
& \quad x_1, x_2 \text{ integer}
\end{align*}
\]
Geom. interpretation, Lagrangian Relaxation

Original problem, integer solution

\[(x_1, x_2) = (4, 0) \quad z = 28.00\]

Original problem, LP-relaxed solution

\[(x_1, x_2) = \left(\frac{36}{11}, \frac{40}{11}\right) = (3.27, 3.64) \quad z = 30.18\]

Drop first constraint, integer solution

\[(x_1, x_2) = (3, 4) \quad z = 29.00\]

Drop first constraint, LP-relaxed solution

\[(x_1, x_2) = \left(\frac{16}{5}, 4\right) = (3.2, 4) \quad z = 30.40\]

Maximum on \(Q\), LP-relaxed solution

\[(x_1, x_2) = (3, 4) \quad z = 29.00\]

Maximum on \(Q\), with first constraint added

\[(x_1, x_2) = \left(\frac{28}{9}, \frac{32}{9}\right) = (3.11, 3.56) \quad z = 28.88\]
Geom. interpretation, Lagrangian Relaxation

Viewpoint 1: fixed $\lambda$

\[
\begin{align*}
\text{max} \quad & (7 + \lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda \\
\text{s.t.} \quad & 5x_1 + x_2 \leq 20 \\
& -2x_1 - 2x_2 \leq -7 \\
& -x_1 \leq -2 \\
& x_2 \leq 4 \\
& x_1, x_2 \text{ integer}
\end{align*}
\]

Redefinition using convex hull of $Q$

\[
\begin{align*}
\text{max} \quad & (7 + \lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda \\
\text{s.t.} \quad & 4x_1 + x_2 \leq 16 \\
& -x_1 - x_2 \leq -4 \\
& -x_1 \leq -2 \\
& x_2 \leq 4
\end{align*}
\]
**Geom. interpretation, Lagrangian Relaxation**

Viewpoint 1:

- $\lambda$ is a modifier of the objective function
- For $0 \leq \lambda \leq \frac{1}{9}$, optimal solution $(3, 4)$
  \[ z_{LR}(\lambda) = (7 + \lambda)3 + (2 - 2\lambda)4 + 4\lambda = 29 - \lambda \]
- For $\lambda \geq \frac{1}{9}$ optimal solution $(4, 0)$
  \[ z_{LR}(\lambda) = (7 + \lambda)4 + (2 - 2\lambda)0 + 4\lambda = 28 + 8\lambda \]
- Increasing lambda is forcing the optimal solution to satisfy relaxed constraint.
• When $\lambda = \frac{1}{9}$ we get the tightest bound.
• In this case the isoprofit line is parallel to the line through $(3, 4)$ and $(4, 0)$.
• We may choose an arbitrary point $x^*$ on this line
  \[ (x^*_1, x^*_2) = \left( \frac{28}{9}, \frac{32}{9} \right) = (3.11, 3.56) \]
  which satisfies the relaxed constraint
  \[ -x_1 + 2x_2 \leq 4 \]
• In this case
  \[ z_{LD} = \max \{ cx : Dx \leq d, x \in \text{conv}(Q) \} \]
This “proves” theorem 10.3 page 172.
Geom. interpretation, Lagrangian Relaxation

Integer Programming Problem

\[
\begin{align*}
\text{maximize} & \quad cx \\
\text{subject to} & \quad Ax \leq b \\
& \quad Dx \leq d \\
& \quad x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

\[
\max \left\{ cx : x \in \text{conv}(Ax \leq b, Dx \leq d, x \in \mathbb{Z}_+) \right\}
\]

Lagrange Relaxation, multipliers \( \lambda \geq 0 \)

\[
\begin{align*}
\text{maximize} & \quad z_{LR}(\lambda) = cx - \lambda(Dx - d) \\
\text{subject to} & \quad Ax \leq b \\
& \quad x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

for best multiplier \( \lambda \geq 0 \)

\[
\max \left\{ cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+) \right\}
\]
Geom. interpretation, Lagrangian Relaxation

Viewpoint 2: fixed point $x^i$

There are 8 integer points in $Q$:

$$\{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8\} = \{(2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,0)\}$$

For fixed $x^i$ the objective function

$$z_{LR}(\lambda, x^i) = (7 + \lambda)x_1^i + (2 - 2\lambda)x_2^i + 4\lambda = 7x_1^i + 2x_2^i + \lambda(x_1^i - 2x_2^i + 4)$$

is an affine function.

E.g. for $x^7 = (3, 4)$

$$z_{LR}(\lambda, x^7) = 7 \cdot 3 + 2 \cdot 4 + \lambda(3 - 2 \cdot 4 + 4) = 29 - \lambda$$
Geom. interpretation, Lagrangian Relaxation

Viewpoint 2:

Objective

\[ z_{LR}(\lambda) = \max_{x^i \in Q} z(\lambda, x^i) \]

**Proposition 2** The lagrangian relaxed problem \( z_{LR}(\lambda) \) as function of the multipliers \( \lambda \in \mathbb{R}, \lambda \geq 0 \) is piecewise linear and convex

(see Wolsey, figure page 173)
Lagrangian relaxation and duality

- Lagrangian relaxation is a generalization of duality, where we may relax any subset of constraints.

- Lagrange Relaxation

  $$\begin{align*}
  \text{maximize} \quad & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\
  \text{subject to} \quad & Ax \leq b \\
  & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
  \end{align*}$$

Lagrangian Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

is an LP-problem

- Optimal multipliers $\lambda$ may be found by simplex.

- Subgradient is however faster when few iterations.
Lagrangian Relaxation

Integer Programming Problem

\[
\begin{align*}
\text{maximize} & \quad cx \\
\text{subject to} & \quad Ax \leq b \\
& \quad Dx \leq d \\
& \quad x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

Lagrange Relaxation, multipliers \( \lambda \geq 0 \)

\[
\begin{align*}
\text{maximize} & \quad z_{LR}(\lambda) = cx - \lambda(Dx - d) \\
\text{subject to} & \quad Ax \leq b \\
& \quad x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n
\end{align*}
\]

Lagrangian Dual Problem

\[
z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)
\]

Assume that the “nice constraints” \( Ax \leq b \) define the convex hull, e.g.

- \( A \) is totally unimodular, and \( b \) is a vector of integers
- There are no constraints left
- The remaining constraints are defined in linear variables
Lagrangian Relaxation

for best multiplier $\lambda \geq 0$ strength of model

$$\max \left\{ cx : Dx \leq d, x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+) \right\}$$

If $\{x : Ax \leq b\} = \{x \in \text{conv}(Ax \leq b, x \in \mathbb{Z}_+)\}$ strength

$$\max \left\{ cx : Dx \leq d, Ax \leq b \right\}$$

Corollary (page 173 in Wolsey)

$$z_{LD} = z_{LP}$$

for any objective function $cx$.

- We do not obtain better bounds than by linear relaxation.
- We may find $z_{LP} = z_{LD}$ in polynomial time.
- If the remaining problem $Ax \leq b$ has a nice structure (e.g. min-spanning-tree) we may find $z_{LD}$ faster than $z_{LP}$.  

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Lagrangian Relaxation

How should we choose the optimal multipliers $\lambda$ (i.e. the multipliers which minimize $z_{LD}$) in this case

Consider $z_{LP}$ as solution to

$$
\begin{align*}
\text{maximize} & \quad cx \\
\text{subject to} & \quad Ax \leq b \\
& \quad A'x \leq b' \\
& \quad x \geq 0
\end{align*}
$$

where the dual problem is

$$
\begin{align*}
\text{minimize} & \quad by + b'y' \\
\text{subject to} & \quad yA + y'A' \geq c \\
& \quad y, y' \geq 0
\end{align*}
$$

Now consider the lagrangian relaxed problem $z_{LR}(\lambda)$

$$
\begin{align*}
\lambda b' & + \text{maximize} \quad (c - \lambda A')x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0 \quad \text{(IP sol. for free)}
\end{align*}
$$

where the dual problem is

$$
\begin{align*}
\lambda b' & + \text{minimize} \quad by \\
\text{subject to} & \quad yA \geq c - \lambda A' \\
& \quad \lambda, y \geq 0
\end{align*}
$$
Lagrangian Relaxation

Thus

- If relax all constraints: ordinary dual problem
- Lagrangian relaxation of a constraint can be seen as “dualization” of a constraint.
- We have found a technique for deriving the best lagrangian multipliers in some special cases.