Wednesday, November 19

Program of the day:

● Repetition: Definition of facets, dimension
● Cover inequalities
● Separation of valid inequalities
● Lifting of inequalities
● Applications: The Traveling Salesman Problem, separation of subtour constraints
● “Projektopgave P2”
**Dominance**

maximize \( \ldots \)  
subject to  
\[
\begin{align*}
1x_1 + 3x_2 & \leq 4 \\
2x_1 + 4x_2 & \leq 9 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Multiplying the second inequality with \( u = \frac{1}{2} \)

\[
1x_1 + 2x_2 \leq \frac{9}{2}
\]

First inequality dominates the second.

**Dominance:**

\[
\begin{align*}
\pi x & \leq \pi_0 \\
\mu x & \leq \mu_0
\end{align*}
\]

\( \pi x \leq \pi_0 \) dominates \( \mu x \leq \mu_0 \) if there exists \( u > 0 \) such that \( \pi \geq u\mu \) and \( \pi_0 \leq u\mu_0 \).

\[
\begin{align*}
2x_1 + 4x_2 & \leq 9 \\
x_1 + 3x_2 & \leq 4
\end{align*}
\]
Redundance

maximize \[ \ldots \]
subject to \[ 6x_1 - x_2 \leq 9 \]
\[ 9x_1 - 5x_2 \leq 6 \]
\[ 5x_1 - 2x_2 \leq 6 \]
\[ x_1, x_2 \geq 0 \]

Multiplying the first two constraints with \( u = (\frac{1}{3}, \frac{1}{3}) \)
\[ 5x_1 - 2x_2 \leq 5 \]

Last inequality is redundant

Redundance:
\[ \pi^i x \leq \pi^i_0, \quad i = 1, \ldots, k \]
\[ \mu x \leq \mu_0 \]

Inequality \( \mu x \leq \mu_0 \) is redundant if there exists a vector \((u_1, \ldots, u_k) \geq 0\) such that
\[ \left( \sum_{i=1}^{k} u_i \pi^i \right)x_i \leq \sum_{i=1}^{k} u_i \pi^i_0 \]

dominates \( \mu x \leq \mu_0 \)
Polyhedra, Facets

Polyhedra $P \subset \mathbb{R}^2$

subject to $x_1 < 2$
$x_1 + x_2 < 4$
$x_1 + 2x_2 < 10$
$x_1 + 2x_2 \leq 6$
$x_1 + x_2 \leq 2$
$x_1, x_2 \geq 0$

- $P \subset \mathbb{R}^2$ and “both directions are present”
- $P$ is full-dimensional.
- The points $(2,0), (1,1)$ and $(2,2)$ are affinely independant points.
- The vectors $(2,0,1), (1,1,1)$ and $(2,2,1)$ are linearly independant.
- The dimension of $P$ is one less than the number of affinely independant points.
Polyhedra, Facets

Affinely independant

The points $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ are affinely independant if the $k - 1$ directions $(x^2 - x^1), \ldots, (x^k - x^1)$ are linearly independant.

Dimension

The dimension of $P$, denoted $\dim(P)$ is one less than the maximum number of affinely independant points in $P$.

Full-dimensional

The polyhedra $P \subseteq \mathbb{R}^n$ is full-dimensional if and only if $\dim(P) = n$.

Face

If $\pi x \leq \pi_0$ is a valid inequality of $P$ then $F$ is a face of $P$ $F = \{x \in P : \pi x = \pi_0\}$

Facet

$F$ is a facet of $P$ iff
- $F$ is a face of $P$
- $\dim(F) = \dim(P) - 1$
**IP-problems**

\[ P = \{ x : Ax \leq b \} \cap \mathbb{Z}^2 \]

- Dimension of \( P \) is 2
- The facet defining inequality must be valid
- A facet should have dimension 1
- There should be 2 affine indendent points on a facet
Cover inequalities

\[ 11x_1 + 6x_2 - 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 13 \]
\[ x \in \{0, 1\} \]

To get positive coefficients we substitute \( x_3 = 1 - x'_3 \)

\[ 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \]

Observe

- At most two of \( x_1, x_2 \) and \( x_3 \) can be 1.
- At most two of \( x_1, x_2 \) and \( x_6 \) can be 1.
- At most two of \( x_1, x_5 \) and \( x_6 \) can be 1.
- At most three of \( x_3, x_4, x_5 \) and \( x_6 \) can be 1.
Cover inequalities

Consider the set

\[
X = \left\{ x \in \mathbb{B}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}
\]

We assume that \( a_j \geq 0 \) and \( b \geq 0 \).

Cover

A set \( C \subseteq N \) is a cover if

\[
\sum_{j \in C} a_j > b
\]

A set \( C \subseteq N \) is a minimal cover if \( C \setminus \{j\} \) is not a cover for any \( j \in C \)

Cover Inequality

If \( C \) is a cover the cover inequality

\[
\sum_{j \in C} x_j \leq |C| - 1
\]

is valid for \( X \).
Cover inequalities

\[ 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \]

Minimal cover inequalities

\[
\begin{align*}
&x_1 + x_2 + x_3 \leq 2 \\
&x_1 + x_2 + x_6 \leq 2 \\
&x_1 + x_5 + x_6 \leq 2 \\
&x_3 + x_4 + x_5 + x_6 \leq 3
\end{align*}
\]

Extended cover inequalities for \( C = \{3, 4, 5, 6\} \)

\[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3 \]

which dominates

\[ x_3 + x_4 + x_5 + x_6 \leq 3 \]

Extended cover inequalities

If \( C \) is a cover for \( X \), the extended cover inequality

\[
\sum_{j \in E(C)} x_j \leq |C| - 1
\]

is valid, where

\[ E(C) = C \cup \{j \in N : a_j \geq a_i \text{ for all } i \in C\} \]
Lifting Cover Inequalities

\[ 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \quad (1) \]

Have found cover inequality

\[ x_3 + x_4 + x_5 + x_6 \leq 3 \]

What is the value of \( \alpha_1 \) such that

\[ \alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3 \quad (2) \]

is valid for all \( x \in X \)?

Constraint (2) must be valid whenever (1) is valid. Most difficult to satisfy (2) when \( x_2 = x_7 = 0 \).

- \( x_1 = 0 \) then (2) is valid.
- \( x_1 = 1 \) then we demand that

\[ \alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3 \]

whenever

\[ 11 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 \]

Maximize \( x_3 + x_4 + x_5 + x_6 \) subject to the second inequality

\[ \gamma = \text{maximize } x_3 + x_4 + x_5 + x_6 \]

subject to \( 11 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 \)

\[ x_i \in \{0, 1\} \]

a Knapsack Problem with solution \( \gamma = 1 \).

Thus \( \alpha_1 = 3 - \gamma = 2. \)
Separation problem

The separation problem decides whether a LP-solution vector satisfies all constraints of a given family $\mathcal{F}$. If it does not, it must return a violated constraint in $\mathcal{F}$.
Separation of cover inequalities

We consider a large IP model with $a_{ij} \in \mathbb{Z}$

$$\begin{align*}
\text{maximize} & \quad cx \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \in \{0, 1\}
\end{align*}$$

Current solution $x = x'$ is fractional.
Pick a constraint

$$\sum_{i \in I} a_i x_i \leq b$$

Solve problem

$$\gamma = \min \sum_{i \in I} (1 - x'_i) \delta_i$$

subject to

$$\sum_{i \in I} a_i \delta_i \geq b + 1 \quad (3)$$

$$\delta_i \in \{0, 1\}, \quad i \in I.$$ 

If $\gamma < 1$, let

$$C = \{i \in I : \delta_i = 1\}$$

New inequality

$$\sum_{i \in C} x_i \leq |C| - 1 \quad (4)$$
**C is a minimal cover**

- It is a *cover* since
  \[
  \sum_{i \in C} a_i = \sum_{i \in I} a_i x_i \geq b + 1 > b
  \]

- It is a *minimal* cover, since if we were able to remove an item \( j \) from \( C \) and still have a cover, then we would have a solution to (3) with smaller objective function.

- It is a *violated* inequality since assume that (4) actually was satisfied for the current solution \( x' \). Then we can choose \( \delta_k = 1 \) for \( k \in C \) as a solution to (3). This is a valid solution (due to the definition of \( C \)), and it has objective value
  \[
  \sum_{i \in I} (1 - x'_i) \delta_i = \sum_{i \in C} (1 - x'_i) = |C| - \sum_{i \in C} x'_i \geq |C| - |C| + 1 = 1
  \]

  But this violates the assumption saying that \( \gamma < 1 \).
Symmetric Traveling Salesman Problem

One of most famous and most applicable optimization problems

Given a finite number of ”cities” along with the cost of travel between each pair of them, find the cheapest way of visiting all the cities and returning to your starting point.

Recently Applegate, Bixby, Chvatal, Cook solved USA13509 and Germany15112
Traveling Salesman Problem

- Set of $V$ cities
- To each edge $e \in E$ is associated a cost $c_e$
- Visit each city exactly once
- Minimize travel cost
- $x_e = 1$ if edge $e$ is used

Model 1

$$\min \sum_{e \in E} c_e x_e$$

s.t.  
$$\sum_{e \in \delta(j)} x_e = 2 \quad , \quad j \in V$$
$$\sum_{e \in E(S)} x_e \leq |S| - 1 \quad , \quad S \subset V, S \neq V$$
$$x_e \in \{0, 1\}$$

Model 2

$$\min \sum_{e \in E} c_e x_e$$

s.t.  
$$\sum_{e \in \delta(j)} x_e = 2 \quad , \quad j \in V$$
$$\sum_{e \in \delta(S)} x_e \geq 2 \quad , \quad S \subset V, S \neq V$$
$$x_e \in \{0, 1\}$$

degree constraint, subtour elimination constraint
Traveling Salesman Problem

Subtour LP
\[
\begin{align*}
\min & \quad \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(j)} x_e = 2, \ j \in V \\
& \quad \sum_{e \in \delta(S)} x_e \geq 2, \ S \subset V, S \neq V \\
& \quad 0 \leq x_e \leq 1
\end{align*}
\]

exponentially many constraints

Cutting plane algorithm:
1. solve problem without subtour elimination constraints getting \( x_e^* \)
2. if \( x_e^* \) is a hamilton cycle, stop
3. solve separation problem obtaining a valid inequality
\[
\sum_{e \in \delta(S)} x_e \geq 2
\]
such that
\[
\sum_{e \in \delta(S)} x_e^* < 2
\]
4. add the valid inequality to the problem and repeat
Traveling Salesman Problem

Separation problem:

- capacitated network \((V,E,d)\)
- \(d_e = x_e^*\)
- find min cut in graph
- optimal solution has value less than 2 iff violated constraint exists
- min-cut can be found in \(O(nm \log n)\) time where \(n = |V|\) and \(m = |E|\).
- try each pair of nodes, i.e. run min-cut \(n(n - 1)/2\) times

\[
c_e = \begin{pmatrix}
- & 4 & 3 & 3 & 5 & 2 & 5 \\
- & - & 5 & 3 & 3 & 4 & 7 \\
- & - & - & 4 & 6 & 0 & 4 \\
- & - & - & - & 4 & 4 & 6 \\
- & - & - & - & - & 5 & 8 \\
- & - & - & - & - & - & 3 \\
- & - & - & - & - & - & - \\
\end{pmatrix}
\]
Traveling Salesman Problem

minimize
+ 4 x_{12} + 3 x_{13} + 3 x_{14} + 5 x_{15} + 2 x_{16} + 5 x_{17}
+ 5 x_{23} + 3 x_{24} + 3 x_{25} + 4 x_{26} + 7 x_{27}
+ 4 x_{34} + 6 x_{35} + 0 x_{36} + 4 x_{37}
+ 4 x_{45} + 4 x_{46} + 6 x_{47}
+ 5 x_{56} + 8 x_{57}
+ 3 x_{67}

subject to
\begin{align*}
x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} &= 2 \\
x_{12} &+ x_{23} + x_{24} + x_{25} + x_{26} + x_{27} = 2 \\
x_{13} + x_{23} &+ x_{34} + x_{35} + x_{36} + x_{37} = 2 \\
x_{14} + x_{24} + x_{34} &+ x_{45} + x_{46} + x_{47} = 2 \\
x_{15} + x_{25} + x_{35} &+ x_{45} + x_{56} + x_{57} = 2 \\
x_{16} + x_{26} + x_{36} &+ x_{46} + x_{56} + x_{67} = 2 \\
x_{17} + x_{27} + x_{37} &+ x_{47} + x_{57} + x_{67} = 2 \\
\end{align*}

binary
\begin{align*}
x_{12} & x_{13} x_{14} x_{15} x_{16} x_{17} \\
x_{23} & x_{24} x_{25} x_{26} x_{27} \\
x_{34} & x_{35} x_{36} x_{37} \\
x_{45} & x_{46} x_{47} \\
x_{56} & x_{57} \\
x_{67} &
\end{align*}

end
Traveling Salesman Problem

minimize
\[ + 4 \ x_{12} + 3 \ x_{13} + 3 \ x_{14} + 5 \ x_{15} + 2 \ x_{16} + 5 \ x_{17} + 5 \ x_{23} + 3 \ x_{24} + 3 \ x_{25} + 4 \ x_{26} + 7 \ x_{27} + 4 \ x_{34} + 6 \ x_{35} + 0 \ x_{36} + 4 \ x_{37} + 4 \ x_{45} + 4 \ x_{46} + 6 \ x_{47} + 5 \ x_{56} + 8 \ x_{57} + 3 \ x_{67} \]

subject to
\[ x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} = 2 \]
\[ x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} = 2 \]
\[ x_{13} + x_{23} + x_{34} + x_{35} + x_{36} + x_{37} = 2 \]
\[ x_{14} + x_{24} + x_{34} + x_{45} + x_{46} + x_{47} = 2 \]
\[ x_{15} + x_{25} + x_{35} + x_{45} + x_{56} + x_{57} = 2 \]
\[ x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{67} = 2 \]
\[ x_{17} + x_{27} + x_{37} + x_{47} + x_{57} + x_{67} = 2 \]
\[ x_{13} + x_{23} + x_{34} + x_{35} + x_{16} + x_{26} + x_{46} + x_{56} + x_{17} + x_{27} + x_{47} + x_{57} >= 2 \]

binary
\[ x_{12} \ x_{13} \ x_{14} \ x_{15} \ x_{16} \ x_{17} \]
\[ x_{23} \ x_{24} \ x_{25} \ x_{26} \ x_{27} \]
\[ x_{34} \ x_{35} \ x_{36} \ x_{37} \]
\[ x_{45} \ x_{46} \ x_{47} \]
\[ x_{56} \ x_{57} \]
\[ x_{67} \]

end
Traveling Salesman Problem

minimize
\[ + 4 \ x_{12} + 3 \ x_{13} + 3 \ x_{14} + 5 \ x_{15} + 2 \ x_{16} + 5 \ x_{17} \]
\[ + 5 \ x_{23} + 3 \ x_{24} + 3 \ x_{25} + 4 \ x_{26} + 7 \ x_{27} \]
\[ + 4 \ x_{34} + 6 \ x_{35} + 0 \ x_{36} + 4 \ x_{37} \]
\[ + 4 \ x_{45} + 4 \ x_{46} + 6 \ x_{47} \]
\[ + 5 \ x_{56} + 8 \ x_{57} \]
\[ + 3 \ x_{67} \]

subject to
\[ x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} = 2 \]
\[ x_{12} + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} = 2 \]
\[ x_{13} + x_{23} + x_{34} + x_{35} + x_{36} + x_{37} = 2 \]
\[ x_{14} + x_{24} + x_{34} + x_{45} + x_{46} + x_{47} = 2 \]
\[ x_{15} + x_{25} + x_{35} + x_{45} + x_{56} + x_{57} = 2 \]
\[ x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{67} = 2 \]
\[ x_{17} + x_{27} + x_{37} + x_{47} + x_{57} + x_{67} = 2 \]

\[ x_{13} + x_{23} + x_{34} + x_{35} + \]
\[ x_{16} + x_{26} + x_{46} + x_{56} + \]
\[ x_{17} + x_{27} + x_{47} + x_{57} \geq 2 \]

\[ x_{12} + x_{23} + x_{26} + x_{27} + \]
\[ x_{14} + x_{34} + x_{46} + x_{67} + \]
\[ x_{15} + x_{35} + x_{56} + x_{57} \geq 2 \]

binary
\[ x_{12} \ x_{13} \ x_{14} \ x_{15} \ x_{16} \ x_{17} \]
\[ x_{23} \ x_{24} \ x_{25} \ x_{26} \ x_{27} \]
\[ x_{34} \ x_{35} \ x_{36} \ x_{37} \]
\[ x_{45} \ x_{46} \ x_{47} \]
\[ x_{56} \ x_{57} \]
\[ x_{67} \]

end