SOLUTION

A single figure is better than a thousand words! Thus, to facilitate the overview we shall consider the planning problem at hand as a network flow problem with

\[ n \text{ sources} \quad A_1, \ldots, A_n \quad \text{(production)} \]
\[ n \text{ destinations} \quad B_1, \ldots, B_n \quad \text{(demands)} \]

and \( n \) intermediate vertices \( I_1, \ldots, I_n \).

The corresponding directed network becomes

\[ \text{sources} \]
\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \]
\[ A_1 \quad A_2 \quad A_i \quad A_n \]
\[ z_1 \quad z_2 \quad z_i \quad z_n \]

\[ \text{intermed.} \]
\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \]
\[ I_1 \quad I_2 \quad I_i \quad I_n \]
\[ d_1 \quad d_2 \quad d_i \quad d_n \]

\[ \text{destinations} \]
\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \]
\[ B_1 \quad B_2 \quad B_i \quad B_n \]

M1: The objective is to satisfy total demand at least total cost. For week \( i, i = 1, \ldots, n \), the production costs are \( p_i z_i \) and the cost of keeping \( s_i \) units in stock from month \( i \) to month \( i+1 \) is \( r_i s_i \). These are the two constituents of the objective function as correctly reflected in answer c).

M2: Additional constraints? A flow conservation equation must apply for each of the intermediate vertices,

Vertex \( I_i \):

\[ \text{inflow} = s_{i-1} + z_i = s_i + d_i = \text{outflow} \]

or

\[ s_{i-1} + z_i - s_i = d_i, \quad i = 1, \ldots, n \]

as in answer d). Note that \( z_i \leq d_i + s_i \) (answer a)) is necessary but not sufficient. a) is automatically satisfied by all solutions satisfying d) and is therefore redundant.
M3: For the given instance, the corresponding network is

```
   2   5   12
  /   /   /
 /   /   /
3   6   d_1=0 d_2=0 d_3=8
```

Let $q_{ij}$ be the per unit cost of supplying the demand in week $j$ from the production set up in week $i$, $i \leq j$. We find: $q_{13} = 2+3+6 = 11$, $q_{23} = 5+6 = 11$, $q_{33} = 12$. The correct answer is thus $d_3 \times \min(q_{13}, q_{23}, q_{33}) = 8 \times 11 = 88$ as in answer c).

This question is actually meant to show that the entire problem is decomposable in general and optimally solvable "by inspection": nothing more is needed than repeat the above argument for each week having a positive demand.

M4: (1) is indeed true since we have $n$ linearly independent constraints, cf. M2. It follows then that (3) is true as well whereas (2) is sheer nonsense. Hence, c) is correct.

M5: Since $c_q$ must equal $q_{i,j}d_p$, we find for all $i, j, < j$:

<p>| | | | | | |</p>
<table>
<thead>
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<tbody>
<tr>
<td>$n = 4$</td>
<td>$d_1 = 7$, $r_1 = 3$</td>
<td>$d_2 = 9$, $r_2 = 6$</td>
<td>$d_3 = 8$, $r_3 = 4$</td>
<td>$d_4 = 5$</td>
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<tr>
<td>$p_1 = 2$</td>
<td>$c_{11} = (2+0)7 = 14$</td>
<td>$c_{12} = (2+3)9 = 45$</td>
<td>$c_{13} = (2+3+6)8 = 88$</td>
<td>$c_{14} = (2+3+6+4)5 = 75$</td>
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<tr>
<td>$p_2 = 5$</td>
<td>$c_{21} =$</td>
<td>$c_{22} = (5+0)9 = 45$</td>
<td>$c_{23} = (5+6)8 = 88$</td>
<td>$c_{24} = (5+6+4)5 = 75$</td>
<td></td>
</tr>
<tr>
<td>$p_3 = 12$</td>
<td>$c_{31} =$</td>
<td>$c_{32} =$</td>
<td>$c_{33} = (12+0)8 = 96$</td>
<td>$c_{34} = (12+4)5 = 80$</td>
<td></td>
</tr>
<tr>
<td>$p_4 = 14$</td>
<td>$c_{41} =$</td>
<td>$c_{42} =$</td>
<td>$c_{43} =$</td>
<td>$c_{44} = (14+0)5 = 70$</td>
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Thus, $c_{14} + c_{24} + c_{34} + c_{44} = 75+75+80+70 = 300$ as in answer c). Note that the correct value of the sum must be divisible by $d_4 = 5$. A clever shortcut is therefore to realize that "300" is the only number among those listed satisfying that property.

T1: i) The optimal solution(s) can be derived from the entries in the C-matrix representing the column mimima as shown in bold below:

```
```
The minimum value of the objective function is $14+45+88+70 = 217$.

Alternatively, LP* can be viewed as a balanced Transportation Problem:

The numbers above are self-explanatory and it is easily verified that the same optimal solution obtains.

ii) It is seen that $z_3 = 0$ and $z_4 = 5$ in all optimal solutions. As to $z_1$ and $z_2$ there are 4 alternate solutions, all reflecting the decomposition principle used:

\[
\begin{array}{cccc}
| & 7 & 16 & 15 & 24 \\
\hline
z_1 & & & & \\
\hline
z_2 & 17 & 8 & 9 & 0 \\
\end{array}
\]

However, if that principle is abandoned, we have in total 18 alternate solutions:

$$z_1 = 7+\delta, \quad z_2 = 17-\delta, \quad \delta = 0, 1, \ldots, 17$$

Finally, if "fractional" devices IT03 are allowed to be manufactured and kept in stock, we have infinitely many alternate solutions:

$$z_1 = 7+\delta, \quad z_2 = 17-\delta, \quad 0 \leq \delta \leq 17$$

iii) If $p_4 = 14$ is replaced by $p_4 = 14+\Delta$, only the "$c_{44}$ entry" is affected. It appears that the optimal solution remains optimal for $\Delta<1$. For $\Delta=1$, the demand $d_4$ can be
satisfied from the production made in weeks 1, 2, 4 at a unit cost of 15. For \( \Delta > 1 \) it is no longer profitable to set up a production in week 4.

**T2:**

**LP-Y**, shown below with the constraints numbered by [1]-[7], appears to optimally solvable by inspection

\[
\begin{align*}
\text{max} & \quad 7y_1 + 9y_2 + 8y_3 + 5y_4 \\
y_1 & \leq 2 \quad [1] \\
y_2 & \leq 5 \quad [2] \\
y_3 & \leq 12 \quad [3] \\
y_4 & \leq 14 \quad [4] \\
-y_1 + y_2 & \leq 3 \quad [5] \\
-y_2 + y_3 & \leq 6 \quad [6] \\
-y_3 + y_4 & \leq 4 \quad [7]
\end{align*}
\]

\( y_1, y_2, y_3, y_4 \) free

Since all variables have positive coefficients in the objective function, a first idea is to make all variables as large as possible. Disregarding the upper bounds on each of the 4 variables (constraints [1]-[4]) the structure of the remaining 3 constraints supports that idea: the larger \( y_k \) is, the larger value can be assigned to \( y_{k+1}, k=1,2,3 \). Hence, an optimal solution is

\[
\begin{align*}
y_1 &= 2 \quad [1], \\
y_2 &\leq 5 \quad [2], \quad y_2 \leq 3 + y_1 = 5 \quad [6] \\
\Rightarrow & \quad y_2 = 5, \\
y_3 &\leq 12 \quad [3], \quad y_3 \leq 6 + y_2 = 11 \quad [6] \\
\Rightarrow & \quad y_3 = 11, \\
y_4 &\leq 14 \quad [4], \quad y_4 \leq 4 + y_3 = 15 \quad [7] \\
\Rightarrow & \quad y_4 = 14
\end{align*}
\]

with a maximum value of the objective function equalling 217.

**T3:**

i) Did **LP-Y** come from a clear sky or is "**T2**, ii)" in any way related to some of the previous questions?

Good clues are the rediscovery of the number "217" and the designation of the 7 variables. With variables \( z_1, z_2, z_3, z_4, s_1, s_2, s_3 \), the dual **LP-ZS** of **LP-Y** reads
\[
\begin{array}{cccccc}
\text{obj.fct.:} & z_1 & z_2 & z_3 & z_4 & s_1 & s_2 & s_3 \\
& 2 & 5 & 12 & 14 & 3 & 6 & 4 & (\text{min}) \\
1 & -1 & 1 & -1 & & & & = 7 \\
1 & 1 & 1 & -1 & & & & = 8 \\
& & & 1 & 1 & & & = 5 \\
\end{array}
\]

All variables nonnegative

which is recognized as the instance called \( \text{LP}^* \), cf. M5.

ii) Already done: see T1 i)!
Answers

- M10:

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5
\end{array}
\]

Rows: vertices 1,2,3,4,5 Columns: edges in order top-down

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Hence the correct answer is c).

- T6: We prove the stated by use of proposition 3.2 page 39 in Wolsey. The partitioning of the columns into two sets \(M_1, M_2\) is chosen as the partitioning of the vertices in the bipartite graph.

- M11: Adding slack variables \(s_1, s_2 \geq 0\) we get the model:

\[
\begin{align*}
\text{maximize} & \quad 4x_1 + 2x_2 + x_3 \\
\text{subject to} & \quad 14x_1 + 10x_2 + 11x_3 + s_1 = 32 \\
& \quad -10x_1 + 8x_2 + 9x_3 - s_2 = 0 \\
& \quad x_1, x_2, x_3 \in \mathbb{Z}_+
\end{align*}
\]

To find the simplex tableau we use the equation

\[
Bx_B + Nx_N = b \quad \Leftrightarrow \quad x_B + B^{-1}Nx_N = B^{-1}b
\]

where \(x_B = (x_1, x_2)\), \(x_N = (x_3, s_1, s_2)\) and

\[
B^{-1} = \frac{1}{212} \begin{pmatrix}
8 & -10 \\
10 & 14
\end{pmatrix}, \quad b = \begin{pmatrix}
32 \\
0
\end{pmatrix}, \quad N = \begin{pmatrix}
11 & 1 & 0 \\
9 & 0 & -1
\end{pmatrix}
\]

Inserting the stated we get the simplex tableau

\[
x_1 + \frac{1}{212}(-2x_3 + 8s_1 + 10s_2) = \frac{1}{212}256 \\
x_2 + \frac{1}{212}(236x_3 + 10s_1 - 14s_2) = \frac{1}{212}320
\]
The Gomory cut is derived from the second equation, as it has defines \( x_2 \), hence from the equation

\[
x_2 + \frac{236}{212}x_3 + \frac{10}{212}s_1 + \frac{-14}{212}s_2 = \frac{320}{212}
\]

giving (see the inequality below (8.10) in Wolsey)

\[
\frac{24}{212}x_3 + \frac{10}{212}s_1 + \frac{198}{212}s_2 \geq \frac{108}{212}
\]

which can be reduced to answer f) given by

\[
24x_3 + 10s_1 + 198s_2 \geq 108
\]

- M12: A cover is \( C = \{1, 2, 3\} \) since \( 14 + 10 + 11 > 32 \). This leads to the cover inequality

\[
x_1 + x_2 + x_3 \leq 2
\]

i.e. answer b).

- T7: It is trivial to show that BIN-PACKING is in \( \mathcal{NP} \). As a certificate we choose the sets \( S_1, S_2, \ldots, S_k \). First we check that every weight \( w_j \in M \) is in exactly one set \( S_i \) (run through sets \( S_i \) and mark corresponding weights \( w_j \), then check that every weight was marked exactly once). Then, for each set \( S_i \) we check the sum \( \sum_{j \in S_i} \leq b \). Both can be done in linear time.

To show the reduction from TWO-PARTITION, set \( M := N \) and choose \( b = \frac{1}{2} \sum_{j \in N} a_j \). The number of bins is \( k = 2 \). The reduction can be done in linear time.

- M13: The customers serviced by a vehicle \( k \in M \) may not have an overall demand larger than the capacity \( q \) of the vehicle. A customer is visited by vehicle \( k \) if \( \sum_{j \in V} x_{ijk} = 1 \), i.e. if some edge out of the corresponding node \( i \) used. This leads to the following formulation:

\[
\sum_{i \in C} d_i \sum_{j \in V} x_{ijk} \leq q, \quad k \in M
\]

The correct answer is a).

- T8: Formulate a constraint which for a pair of vertices \( i, j \), where vehicle \( k \) drives directly from vertex \( i \) to \( j \), ensures that the service time \( s_{jk} \) at customer \( j \) respects the driving time \( t_{ij} \) from vertex \( i \) to vertex \( j \) in relation to the service time \( s_{ik} \).

\[
s_{ik} \quad t_{ij} \quad s_{jk}
\]
Logically, this can be described:

\[ x_{ijk} = 1 \quad \Rightarrow \quad s_{ik} + t_{ij} \leq s_{jk} \quad i, j \in V, k \in M \]

Since the model should be an IP-model (this is an implicit consequence of the main text "To formulate the model as an IP-model") and the definition of an "integer program" page 3 in Wolsey, the model should be in linear form.

For this purpose we find

\[ K = \text{u.b.} (s_{ik} + t_{ij} - s_{jk}) = 24 \]

since the minimum value of \( s_{jk} \) is 0 and the maximum value of \( s_{ik} + t_{ij} \) is 24 as arrival at the depot should be before 24:00.

This leads to the model

\[ s_{ik} + t_{ij} + Kx_{ijk} \leq K + s_{jk} \quad i, j \in V, k \in M \]

rearranging the terms so variables are at the left side, we get

\[ s_{ik} + t_{ij} + Kx_{ijk} - s_{jk} \leq K \quad i, j \in V, k \in M \]

In the literature, the model is frequently written

\[ s_{ik} + t_{ij} - K(1 - x_{ijk}) \leq s_{jk} \quad i, j \in V, k \in M \]