DatV Introduction to optimization and operations research

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1 Smoked ham

(Chvatal 1.6, adapted from Greene et al. (1957)) A meat packing plant produces 480 hams, 400 pork bellies, and 230 picnic hams every day; each of these products can be sold either fresh or smoked. The total number of hams, bellies, and picnics that can be smoked during a normal working day is 420; in addition up to 250 products can be smoked on overtime at a higher cost. The *net* profits are as follows:

	Fresh	Smoked on regular time	Smoked on overtime
Hams	$8	$14	$11
Bellies	$4	$12	$7
Picnics	$4	$13	$9

For example, the following schedule yields a total net profit of $9.965:

	Fresh	Smoked on regular time	Smoked on overtime
Hams	165	280	35
Bellies	295	70	35
Picnics	55	70	105

The objective is to find the schedule that maximizes the total net profit. Formulate as an LP problem in the standard form.

2 Smoked ham, complementary slackness

(Chvatal 5.4) In problem 1, one of the possible strategies is as follows:

- Smoke all 400 bellies on regular time.
- Smoke 20 picnics on regular time and 210 on overtime.
- Smoke 40 hams on overtime and sell 440 fresh.

Use the theorem of complementary slackness to find out whether this strategy is optimal or not.

3 Smoked ham, sensitivity analysis

(Chvatal 5.6) In the optimal solution to problem 1, all the bellies and picnics are smoked. However, sufficiently drastic changes in the market prices might provide an incentive to change this policy. Assume that the market price of fresh bellies increases by x dollars, while all the other prices remain fixed at their original levels.

a) How large would x have to be in order to make it profitable for the plant to sell fresh bellies?

b) Ask and answer a similar question for picnics.

c) How would the sales of small amounts of fresh bellies and picnics affect the rest of the operation?

d) What precisely do “small amounts” mean in this context?

4 Nonlinear models

In management science we normally operate with linear objectives of the form cx and linear constraints of the form $Ax \leq b$. The nature of a problem may however demand, that a product of some variables is introduced:

$$2x + 4y + xy \leq 10$$

a) Assuming that x and y are binary variables, show how this constraint can be written in linear form. (Hint: introduce a new binary variable δ to indicate the product xy).

b) Assume now that x and y are integer variables, with $0 \leq x \leq 3$ and $0 \leq y \leq 3$. Rewrite the above inequality in linear form. (Hint: Express the variables in binary form, e.g. $x = 2\delta_1 + \delta_2$, where $\delta_1, \delta_2 \in \{0, 1\}$).

5 Airplanes

(Hillier and Lieberman 13.10) An airline company is considering the purchase of new long-, medium-, and short-range jet passenger airplanes. The purchase price would be \$33.500.000 for each long-range plane, \$25.000.000 for each medium-range plane, and \$17.500.000 for each short-range plane. The board of directors has authorized a maximum commitment of \$750.000.000 for these purchases. Regardless of which airplanes are purchased, air travel of all distances is expected to be sufficiently large enough so that these planes would be utilized at essentially maximum capacity. It is estimated that the net annul profit (after subtracting capital recovery costs) would be \$2.100.000 per long-range plane, \$1.500.000 per medium-range plane, and \$1.150.000 per short-range plane.

It is predicted that enough trained pilots will be available to the company to crew 30 new airplanes. If only short-ranged planes were purchased, the maintenance facilities would be able to handle 40 new planes. However, each medium-range plane is equivalent to $1\frac{1}{3}$ short-range planes, and each long-range plane is equivalent to $1\frac{2}{3}$ short-range planes in terms of their use of the maintenance facilities.

The information given here was obtained by a preliminary analysis of the problem. A more detailed analysis will be conduced subsequently. However, using the predicted data as a first approximation, management wishes to know how many planes of each type should be purchased to maximize profit.

a) Formulate the problem as an IP model.

b) Use GAMS or CPLEX to solve this problem.

6 Fashionable professor

(Hillier and Lieberman 13.11) An American professor will be spending a short sabbatical leave at the University of Iceland. She wishes to bring all needed items with her on the airplane. After collecting together the professional items that she must have, she finds that airline regulations on space and weight for checked luggage will severely limit the clothes she can take. (She plans to carry on a warm coat, and the purchase a warm Icelandic sweater upon arrival in Iceland). Clothes under considerations for checked luggage include 3 skirts, 3 slacks, 4 tops, and 3 dresses. The professor wants to maximize the number of outfits she will have in Iceland (including the special dress she will wear on the airplane). Each dress constitutes on outfit. Other outfits consists of a combination of top and either a skirt or slacks. However, certain combinations are not fashionable and so will not qualify as an outfit.

In the following table, the combinations that will make an outfit are marked with an “x”.

		Top				
		1	2	3	4	Icelandic Sweater
	1	x	x			x
Skirt	2	x			x	
	3		x	x	x	x
	1	x		x		
Slacks	2	x	x		x	x
	3			x	x	x

The weight (in grams) and volume (in cubic centimeters) of each item is shown in the following table:

		Weight	Volume
	1	600	5.000
Skirt	2	450	3.500
	3	700	3.000
	1	600	3.500
Slacks	2	550	6.000
	3	500	4.000
	1	350	4.000
Top	2	300	3.500
	3	300	3.000
	4	450	5.000
	1	600	6.000
Dress	2	700	5.000
	3	800	4.000
Total allowed		4.000	32.000

a) Formulate a binary IP model to choose which items of clothing to take.

b) Solve the model to optimality, using GAMS or CPLEX.

7 Product lines

(Hillier and Lieberman 13.12) The research and development division of a company has been developing four possible new product lines. Management must now make a decision as to which of these four products actually will be produced and at what levels. Therefore, they have asked the Operations Research Department to formulate a mathematical programming model to find the most profitable product mix.

A substantial cost is associated with beginning the production of any product, as given in the first row of the following table. The marginal net revenue from each unit product is given in the second row of the table.

	Product			
	1	2	3	4
Startup cost	\$50.000	\$40.000	\$70.000	\$60.000
Marginal revenue	\$70	\$60	\$90	\$80

Let the continuous decision variables x_1, x_2, x_3 and x_4 be the production level of products $1, 2, 3$ and 4, respectively. Management has imposed the following policy constraints on these variables

1. No more than two of the products can be produced.

2. Product 3 or 4 can be produced if product 1 or 2 is produced.

3. One of the constraints $5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6000$ and $4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6000$ must be satisfied.

Introduce binary variables to formulate an MIP model.

(Hint: Let X_i denote whether product i is produced. Then contraint 2 can be written $(X_3 \vee X_4) \Rightarrow (X_1 \vee X_2)$. If X_A denotes when the first inequality in 3 is satisfied, and X_B denotes when the second inequality is satisfied, then the constraint 3 is $X_A \vee X_B$. You may choose $\varepsilon = 0.01$)

8 Model building

(Hillier and Lieberman 13.13) Consider the following mathematical model:

$$\text{minimize} \quad z = x_1 + x_2$$

subject to the restrictions

1. Either $x_1 \geq 3$ or $x_2 \geq 3$.
2. At least one of the following inequalities holds

$$\begin{aligned} &2x_1 + x_2 \geq 7 \\ &x_1 + x_2 \geq 5 \\ &x_1 + 2x_2 \geq 7 \end{aligned}$$

3. $|x_1 - x_2| = 0$, or 3, or 6.
4. $x_1 \geq 0, x_2 \geq 0$

Formulate this problem as an MIP model.

(Hint: The first constraint is an exclusive or. You may choose $\varepsilon = 0.01$ in this exercise).

9 Branch-and-bound

(Hillier and Lieberman 13.16) Consider the following IP problem

$$\begin{aligned} \text{maximize} \quad & z = 5x_1 + x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 \leq 4 \\ & x_1 - x_2 \leq 1 \\ & 4x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0, \text{integers} \end{aligned}$$

a) Solve the problem graphically.

b) Solve the LP-relaxation graphically. Round the solution to the *nearest* integer solution and check whether it is feasible. Then enumerate *all* the rounded solutions (rounding each noninteger value *either* up or down), check them for feasibility, and calculate z for those that are feasible. Are any of these feasible rounded solutions optimal to the IP problem?

c) Use the MIP branch-and-bound algorithm described in Wolsey Section 7.4 to solve this problem. For each subproblem, solve its LP-relaxation *graphically*.

10 Cover inequalities

We consider a large IP model

$$\begin{aligned} \text{maximize} \quad & c\delta \\ \text{subject to} \quad & A\delta \leq b \end{aligned} \qquad (1)$$

where δ is a vector of boolean variables. If we solve the LP-relaxation of (1) we obtain a solution vector δ'. If all δ'_i are integers, we are done. Otherwise assume that some of the δ'_i are fractional. In this case it would be useful to add a cover inequality to the formulation which cuts off the current LP-solution.

Thus consider a single constraint from (1) where some of the variables δ'_i of the LP-solution have fractional values.

$$\sum_{i \in I} a_i \delta_i \leq b \tag{2}$$

To derive a violated cardinality constraint for (2) we solve the problem:

$$\begin{aligned} \gamma = \text{minimize} \quad & \sum_{i \in I} (1 - \delta'_i) x_i \\ \text{subject to} \quad & \sum_{i \in I} a_i x_i \geq b + 1 \\ & x_i \in \{0, 1\}, \qquad i \in I. \end{aligned} \tag{3}$$

which is a knapsack problem in minimization form. Notice that the coefficients satisfy that $(1 - \delta'_i) \geq 0$, and when $(1 - \delta'_i) = 0$ we may fix x_i to zero. If we obtain a solution to (3) where $\gamma < 1$ we may construct a cover inequality as follows. Let

$$C = \{i \in I : x_i = 1\} \tag{4}$$

be the set of items chosen in (3). Then we may add a new inequality

$$\sum_{i \in C} \delta_i \leq |C| - 1 \tag{5}$$

to the formulation (1).

a) Prove that the set C is a *minimal cover*.

b) Prove that the inequality (5) is violated for the current LP-solution δ'.

c) Consider the problem

$$\begin{array}{ll} \text{maximize} & 8\delta_1 + 10\delta_2 + 6\delta_3 + 12\delta_4 + 8\delta_5 \\ \text{subject to} & 4\delta_1 + 6\delta_2 + 2\delta_3 + 8\delta_4 + 4\delta_5 \leq 8 \\ & \delta_i, \in \{0, 1\} \end{array}$$

Solve the LP-relxation of the problem (by hand or using GAMS or CPLEX).

d) Derive a valid cover inequality for this problem using the technique described above.

e) Add the new inequality to the problem, and solve the LP-relaxation using GAMS or CPLEX.

11 Valid inequalities

(Nemhauser and Wolsey II.1.9.1). Let $S = \{x \in Z^2_+ : 4x_1 + x_2 \leq 28,\ x_1 + 4x_2 \leq 27,\ x_1 - x_2 \leq 1\}$. Determine the facets of $\text{conv}(S)$ graphically. Then derive each of the facets of $\text{conv}(S)$ as a Chvatal-Gomory inequality. (Hint: see Wolsey page 119 for definition of Chvatal-Gomory inequalities)

12 Valid inequalities (difficult)

(Nemhauser and Wolsey II.1.9.2). Let $S = \{x \in Z^3_+ : 19x_1 + 28x_2 - 184x_3 = 8\}$. Derive the valid inequality

$$x_1 + x_2 + 5x_3 \geq 8$$

using modular arithmetics.

13 Valid inequalities

(Nemhauser and Wolsey II.1.9.3). For $S = \{x \in B^4 : 9x_1 + 7x_2 - 2x_3 - 3x_4 \leq 12,\ 2x_1 + 5x_2 + 1x_3 - 4x_4 \leq 10\}$ show that

$$4x_1 + 5x_2 - 2x_3 - 4x_4 \leq 12$$

is a valid inequality by disjunctive arguments. (Hint: see Wolsey page 130 for definitions)

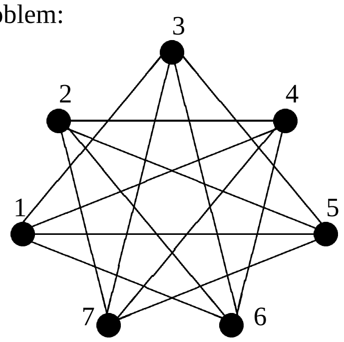
14 Node packing

(Nemhauser and Wolsey II.1.9.4). Given a graph $G = (V, E)$. The *Node packing* problem asks to select the largest number of nodes, such that if node i and j are chosen, then $(i, j) \notin E$.

Let $x_i = 1$ if node i is in packing, $x_i = 0$ otherwise. Then our constraints are:

$$x_i + x_j \leq 1 \text{ for all } (i, j) \in E$$

Consider now the node packing problem:



Let the variable $x_i \in \{0, 1\}$ indicate whether node i is selected. Show that

$$\sum_{i=1}^{7} x_i \leq 2$$

is a valid inequality, both combinatorially and algebraically.

15 Solving node packing problems

a) Formulate the node packing problem of exercise 14 as an LP-problem.

b) Solve the model using GAMS or CPLEX.

c) Add the valid inequality from exercise 14 to the model.

d) Solve the new model using GAMS or CPLEX.

16 Gomory cuts

Consider the problem (the constraints are similar to those in exercise 11).

$$\begin{array}{lrcrcl} \text{maximize} & 7x_1 & + & x_2 & & \\ \text{subject to} & 4x_1 & + & x_2 & \leq & 28 \\ & x_1 & + & 4x_2 & \leq & 27 \\ & x_1 & - & x_2 & \leq & 1 \\ & & & x_1, x_2 & \geq & 0,\ \text{integer} \end{array} \tag{6}$$

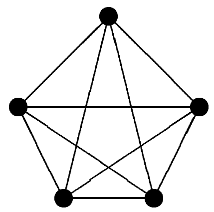
a) Solve the LP-relaxed problem by hand using the simplex algorithm. Remember to add slack variables.

b) Derive a Gomory cut and add it to the problem.

c) Solve the improved formulation by hand or by GAMS or CPLEX.

17 Matching in a graph

A clique K_n in a graph is a complete graph defined on n vertices (i.e. all vertices are connected by an edge). In the following E is the set of edges, and V is the set of vertices.



The matching problem asks for the largest subset of edges, such that no vertex has more than one adjacent edge.

a) Using $x_e \in \{0,1\}$ to indicate whether an edge e is chosen, formulate the matching problem as an optimization problem.

b) Prove that an LP-optimal solution is $x_e = 1/(n-1)$ for all $e \in E$.

c) Assuming that n is odd, derive the cut

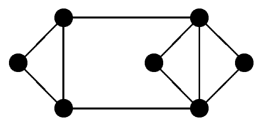
$$\sum_{e \in E} x_e \leq \lfloor n/2 \rfloor$$

d) Is the inequality facet defining?

e) For $n = 5$, formulate and solve the problem to LP-optimality using GAMS or CPLEX. What is the effect of the cut.

18 Node packing

Given a graph $G = (V,E)$ The node packing problem asks to find the largest subset of nodes, such that no two selected nodes are interconnected by an edge.



Let $x_i \in \{0,1\}$ indicate whether a node i is chosen.

a) Write up all valid inequalities for the above graph. Inequalities from cliques as well as odd cycles should be considered.

19 Knapsack Problem

The knapsack problem is a well-known NP-hard optimization problem. Given a number of items j with associated profits p_j and weights w_j, select a subset of these items such that the selected profit is maximized without exceeding a given capacity c. In the present version, we will assume that an item may be selected zero, one or more times. This variant is called the *Unbounded Knapsack Problem* in the literature.

$$\begin{aligned} \max \quad & \textstyle\sum_{j=1}^{n} p_j x_j \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} w_j x_j \leq c \\ & x_j \in \{0,1,\ldots\} \end{aligned} \tag{7}$$

a) A charter tourist would like to bring the following items for his vacations:

item	p_j	w_j
shirt	3	10
trousers	4	11
T-shirt	2	8
shoes	5	14
jacket	4	12

The capacity of the knapsack is $c = 31$. Formulate and solve the problem using GAMS or CPLEX.

20 Column generation

We are considering the cutting stock problem: Given an unlimited number of long tubes of length c. We need to cut some small pieces of length $(w_1, w_2, \ldots, w_m)$. At least b_i pieces of length w_i are demanded. Our objective is to minimize the number of long tubes used. Thus we have the following LP-formulation:

$$\begin{aligned} \min \quad & x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned} \tag{8}$$

where the columns in A are all possible patterns of cutting a single tube (i.e. $\sum_{i=1}^{m} a_{ij} \leq c$ and all $a_{ij} \geq 0$, integer). The solution vector x_i states how many times a given cutting pattern is used in the optimal solution.

Since the number of possible cutting patterns can be large (exponential) we will use delayed column generation to introduce cutting patterns only when needed. For small instances, column generation can be run by hand with a little support by GAMS or CPLEX.

0 Choose an initial solution, where only one kind of pieces are cut from each tube. These cutting patterns form the initial value of A.

1 Solve the system (8) for the current value of A by using GAMS or CPLEX.

2 Print out the dual variables y.

3 Solve the knapsack problem (7) with $p = y$. If the optimal profit sum is smaller than 1 then stop.

4 Add the found knapsack solution x as a new column a_{n+1} in the matrix A and go to step 1.

A plumber (blikkenslager) is repairing the heating system at DIKU. For this purpose he needs to cut some tubes: 4 pieces of length 3, 3 pieces of length 5, and 2 pieces of length 6. All the pieces are cut from some long tubes of length 10.

a) Use column generation to find a lower bound on the number of long tubes needed. Use GAMS or CPLEXto solve the linear problem (8), while subproblems in step 2 and 3 can be solved by using the technique from exercise 19.

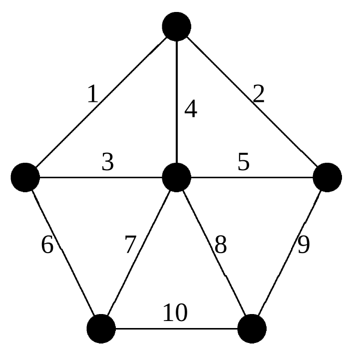
   Compare the found lower bound L_C with the linear bound

$$L_0 = \left\lceil \frac{\sum_{j=1}^{n} w_j}{c} \right\rceil$$

b) Solve the model to integer optimality by any technique you like (guessing is OK). Compare the optimal solution to the two lower bounds derived in question a.

21 Valid inequalities (difficult)

We are solving the traveling salesman problem defined on the following graph



The numbers refer to the edge numbers. We use the boolean variable $x_i \in \{0,1\}$ to determine whether edge i is used in the tour. Prove that the following inequaltiy holds

$$x_1 + x_2 + x_3 + x_4 + x_6 + x_8 \leq 4$$

22 Lagrangian relaxation

(Wolsey section 10, exercise 5) Consider a 0-1 Knapsack Problem

$$\begin{array}{lrcrcrcl} \text{maximize} & 10y_1 & + & 4y_2 & + & 14y_3 & & \\ \text{subject to} & 3y_1 & + & y_2 & + & 4y_3 & \leq & 4 \\ & \multicolumn{7}{l}{y_1, y_2, y_3 \in \{0,1\}} \end{array} \tag{9}$$

a) Construct a lagrangian dual by dualizing the knapsack constraint. What is the optimal value of the dual variable?

b) Suppse one runs the subgradient algorithm using step size (b) in Theorem 10.4 (Wolsey), starting with $u^0 = 0$, $\mu_0 = 1$ and $\rho = \frac{1}{2}$. What is the value of the lagrangian multiplier after 5 iterations of the algorithm.

23 Lagrangian relaxation

(Wolsey section 10, exercise 8) Consider the assignment problem with budget constraint. There are given a set M of employees who should be assigned to the set N of jobs. Each job should be covered by exactly one employee, and every employee should have exactly one job. If employee i is doing the job j a net profit of c_{ij} can be expected. However in order to let employee i do the job j a teaching cost of a_{ij} must be paid. The total teaching costs may not exceed the limit b. The objective is to maximize the net profit. If we introduce the binary variable x_{ij} to denote that employee i gets the job j we get the following formulation

$$\begin{array}{lll} \text{maximize} & \sum_{i\in M}\sum_{j\in N} c_{ij}x_{ij} & \\ \text{subject to} & \sum_{j\in N} x_{ij} = 1 & \text{for} i \in M \\ & \sum_{i\in M} x_{ij} = 1 & \text{for} j \in N \\ & \sum_{i\in M}\sum_{j\in N} a_{ij}x_{ij} \leq b & \\ & x_{ij} \in \{0,1\} & \end{array} \tag{10}$$

Discuss the strength of different possible Lagrangian relaxations, and the ease or difficulty of solving the Lagrangian subproblems, and the Lagrangian dual.

Using boolean variables to model different constraints

logical expression X	IP model	meaning of constraint
$x > 0 \Leftrightarrow \delta = 1$	$x - \varepsilon\delta \geq 0$ $x - M\delta \leq 0$ (M u.b. on x)	$\delta = 1 \Rightarrow x \geq \varepsilon$ $x > 0 \Rightarrow \delta = 1$
$\sum_{j=1}^{n} a_j x_j \leq b \Leftrightarrow \delta = 1$	$\sum_{j=1}^{n} a_j x_j + M\delta \leq M + b$ $\sum_{j=1}^{n} a_j x_j - (m - \varepsilon)\delta \geq b + \varepsilon$ (M u.b. on $\sum_{j=1}^{n} a_j x_j - b$) ($m$ l.b. on $\sum_{j=1}^{n} a_j x_j - b$)	$\delta = 1 \Rightarrow \sum_{j=1}^{n} a_j x_j \leq b$ $\sum_{j=1}^{n} a_j x_j \leq b \Rightarrow \delta = 1$
$\sum_{j=1}^{n} a_j x_j \geq b \Leftrightarrow \delta = 1$	$\sum_{j=1}^{n} a_j x_j + m\delta \geq m + b$ $\sum_{j=1}^{n} a_j x_j - (M + \varepsilon)\delta \leq b - \varepsilon$ (M u.b. on $\sum_{j=1}^{n} a_j x_j - b$) ($m$ l.b. on $\sum_{j=1}^{n} a_j x_j - b$)	$\delta = 1 \Rightarrow \sum_{j=1}^{n} a_j x_j \geq b$ $\sum_{j=1}^{n} a_j x_j \geq b \Rightarrow \delta = 1$
$(\delta_1 = 1 \wedge \delta_2 = 1) \Leftrightarrow \delta = 1$	$\delta_1 + \delta_2 - 2\delta \geq 0$ $\delta_1 + \delta_2 - \delta \leq 1$	$\delta = 1 \Rightarrow \delta_1 = 1 \wedge \delta_2 = 1$ $\delta_1 = 1 \wedge \delta_2 = 1 \Rightarrow \delta = 1$
$(\delta_1 = 1 \vee \delta_2 = 1) \Leftrightarrow \delta = 1$	$\delta_1 + \delta_2 - \delta \geq 0$ $\delta_1 + \delta_2 - 2\delta \leq 0$	$\delta = 1 \Rightarrow \delta_1 = 1 \vee \delta_2 = 1$ $\delta_1 = 1 \vee \delta_2 = 1 \Rightarrow \delta = 1$
$(\delta_1 = 1 \Rightarrow \delta_2 = 1) \Leftrightarrow \delta = 1$	$\delta_1 - \delta_2 + \delta \leq 1$ $\delta_1 - \delta_2 + 2\delta \geq 1$	$\delta = 1 \Rightarrow (\delta_1 = 1 \Rightarrow \delta_2 = 1)$ $(\delta_1 = 1 \Rightarrow \delta_2 = 1) \Rightarrow \delta = 1$
$\neg(\delta_1 = 1) \Leftrightarrow \delta = 1$	$\delta = 1 - \delta_1$	

Logical conditions may be modeled by associating an indicator variable δ_i with every condition X_i such that

$$\delta_i = 1 \Leftrightarrow X_i \text{ is true}$$

In this way we may formulate

$X_1 \vee X_2$	$\delta_1 + \delta_2 \geq 1$
$X_1 \wedge X_2$	$\delta_1 = 1, \delta_2 = 1$
$X_1 \Rightarrow X_2$	$\delta_1 - \delta_2 \leq 0$
$X_1 \Leftrightarrow X_2$	$\delta_1 - \delta_2 = 0$

The equations in the right side of the table can either be added directly to the model, or they can be used to trigger a new indicator variable δ which can be used in other parts of the model.

Notice that in LP and MIP constraints implicitly are linked by an “and”, i.e. all the constraints must be satisfied. This however means that conditions linked by an “and” are much easier to model than those linked by an “or”. In many situations it may be fruitful to rewrite an expression to an equivalent “and” form as illustrated in the following example:

$$(X_1 \vee X_2) \Rightarrow (X_3 \wedge X_4)$$

is equivalent to

$$(X_1 \Rightarrow (X_3 \wedge X_4)) \wedge (X_2 \Rightarrow (X_3 \wedge X_4))$$

or further decomposed to

$$(X_1 \Rightarrow X_3) \wedge (X_1 \Rightarrow X_4) \wedge (X_2 \Rightarrow X_3) \wedge (X_2 \Rightarrow X_4)$$

Introducing an indicator variable δ_i with each of the logical conditions X_i we may easily model this criteria by the constraints

$$\delta_1 - \delta_3 \leq 0, \quad \delta_1 - \delta_4 \leq 0, \quad \delta_2 - \delta_3 \leq 0, \quad \delta_2 - \delta_4 \leq 0$$