Introduction to Optimization:

Written Exam, 15 December 2000

Your assignment

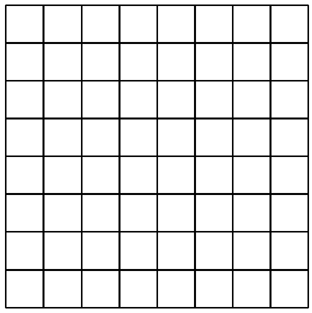
20 different questions Q1-Q20 are posed on the subsequent pages. Q1-Q8 and Q11-Q18 are *multiple choice questions.* For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation to write, for example, "7.b" as your answer to question Q7. Q9-Q10 and Q19-Q20 are ordinary *text questions.* Each correct answer to a multiple choice question gives 4 points, to a text question gives 9 points. The maximum score is thus 100 points.

Note: only the last 10 questions are available

Cutting down the weight — with Gomory cuts

Biotech Ltd. is also producing two pills for slimming. Pill number 1 is a serious product which however has no commercial value. Pill number 2 is a pure plasebo product which due to its pink color is selling very well — and some people even report magnificant results with it. The president of Biotech Ltd. knowns that the "active ingredients" in pill 2 is its very high price which makes people starve to pay the monthly costs. Thus to maximize the production of pill 2 he constructs the following model

$$\begin{array}{llcrcl} \text{maximize} & & & x_2 & & \\ \text{subject to} & 3x_1 & + & 2x_2 & \leq & 6 \\ & -3x_1 & + & 2x_2 & \leq & 0 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0, \text{integer}} \end{array} \qquad (1)$$



Q 11 Classify the four constraints

$$\begin{array}{rl} \text{(i)} & 3x_1 + 2x_2 \leq 6 \\ \text{(ii)} & -3x_1 + 2x_2 \leq 0 \\ \text{(iii)} & x_1 \geq 0 \\ \text{(iv)} & x_2 \geq 0 \end{array}$$

according to their strength

11.a) all constraints (i) to (iv) are facet defining.

11.b) constraint (iii) and (iv) are facet defining.

11.c) constraint (i) is redundant and constraint (iv) is facet defining.

11.d) constraint (iii) is redundant and constraint (iv) is facet defining.

11.e) constraint (iii) and (iv) are redundant.

Q 12 In order to solve the LP-relaxation of the model, slack variables x_3 and x_4 are added to the two constraints. The problem is solved using the simplex algorithm. What is the form of the final simplex tableau

12.a) $\left\{\begin{array}{rcrcrcr} z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \\ x_1 & = & 1 & - & \frac{1}{6}x_3 & + & \frac{1}{6}x_4 \\ x_2 & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \end{array}\right.$

12.b) $\left\{\begin{array}{rcrcrcr} z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \\ x_1 & = & \frac{3}{2} & - & \frac{1}{4}x_3 & + & \frac{1}{4}x_4 \\ x_2 & = & 1 & - & \frac{1}{6}x_3 & - & \frac{1}{6}x_4 \end{array}\right.$

12.c) $\left\{\begin{array}{rcrcrcr} z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \\ x_1 & = & 1 & + & \frac{1}{6}x_3 & + & \frac{1}{6}x_4 \\ x_2 & = & \frac{3}{2} & + & \frac{1}{4}x_3 & + & \frac{1}{4}x_4 \end{array}\right.$

12.d) $\left\{\begin{array}{rcrcrcr} z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \\ x_1 & = & \frac{3}{2} & + & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \\ x_2 & = & 1 & + & \frac{1}{6}x_3 & - & \frac{1}{6}x_4 \end{array}\right.$

Q 13 The first equation $z = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4$ contains a fractional variable, thus a Gomory cut can be derived from this equation. What is the form of the derived inequality if we use the definition of Gomory cuts from Wolsey?

13.a) $\frac{1}{4}x_3 - \frac{1}{4}x_4 \geq -\frac{1}{2}$

13.b) $x_1 \leq 4$

13.c) $x_3 + x_4 \geq 8$

13.d) $x_1 + x_2 \leq 1$

13.e) $x_3 + x_4 \geq 2$

13.f) $x_1 + x_2 \geq 4$

Q 14 Lagrangian relaxing the second constraint $-3x_1 + 2x_2 \leq 0$ in problem (1) leads to a problem LR_λ with only one constraint. What is the objective function of LR_λ

14.a) $x_2 - 3x_1 + \lambda$, where $\lambda \leq 0$

14.b) $(2\lambda + 1)x_2 - 3\lambda x_1$, where $\lambda \geq 0$

14.c) $3\lambda x_1 - 3x_2$, where $\lambda \geq 0$

14.d) $3\lambda x_1 + (1 - 2\lambda)x_2$, where $\lambda \geq 0$

14.e) $3\lambda x_1 - 3x_2$, where $\lambda \leq 0$

14.f) $-3\lambda x_1 - (1 - 2\lambda)x_2$, where $\lambda \leq 0$

Q 15 For which value of λ do we get the tightest upper bound by solving the Lagrangian relaxed problem LR_λ

15.a) $\lambda = 0$

15.b) $\lambda = 1$

15.c) $\lambda = 2$

15.d) $\lambda = \frac{1}{5}$

15.e) $\lambda = \frac{1}{4}$

15.f) $\lambda = \frac{1}{3}$

Uncovering the "Lederhosen" gene

The research lab in Biotech Ltd. is working on a complete classification of the human genes. In particular they are interested in describing the till now unknown "Lederhosen" gene which can be found in certain mountain tribes in Austria. Not surprising this gene can be described by solving a two-constrained knapsack problem:

$$\begin{array}{lrcrcrcrcl} \text{maximize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\ \text{subject to} & 5x_1 & - & 6x_2 & + & 5x_3 & + & 8x_4 & \leq & 5 \\ & 7x_1 & + & 3x_2 & + & 4x_3 & + & 3x_4 & \leq & 9 \\ & x_1 & & & & & & & \leq & 1 \\ & & & x_2 & & & & & \leq & 1 \\ & & & & & x_3 & & & \leq & 1 \\ & & & & & & & x_4 & \leq & 1 \\ & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \geq 0, \text{ integer}} \end{array}$$

Q 16 The LP-relaxation of the problem is solved, giving the dual solution $y_1 = \frac{1}{17}$, $y_2 = \frac{3}{17}$, $y_3 = 0$, $y_4 = \frac{14}{17}$, $y_5 = 0$, $y_6 = 0$. What is the primal solution

16.a) $x_1 = 0$, $x_2 = 1$, $x_3 = \frac{15}{17}$, $x_4 = \frac{14}{17}$

16.b) $x_1 = \frac{15}{17}$, $x_2 = 0$, $x_3 = 1$, $x_4 = \frac{14}{17}$

16.c) $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{1}{17}$, $x_4 = \frac{1}{17}$

16.d) $x_1 = \frac{15}{17}$, $x_2 = \frac{14}{17}$, $x_3 = 0$, $x_4 = 1$

16.e) $x_1 = \frac{14}{17}$, $x_2 = \frac{15}{17}$, $x_3 = 1$, $x_4 = 0$

Q 17 Separate the most violated cover inequality for the first constraint

$$5x_1 - 6x_2 + 5x_3 + 8x_4 \leq 5$$

17.a) $x_1 + x_2 + x_3 \leq 2$

17.b) $x_1 + x_2 + x_3 \leq 1$

17.c) $x_1 - x_2 + x_3 + x_4 \leq 3$

17.d) $x_3 + x_4 \leq 1$

17.e) $x_1 + x_2 + x_3 + x_4 \leq 2$

17.f) $x_4 - x_2 \leq 0$.

(Hint: be careful when solving the separation problem since not all the coefficients are positive)

Q 18 Consider now the second inequality

$$7x_1 + 3x_2 + 4x_3 + 3x_4 \leq 9$$

It is obvious that $C = \{2, 3, 4\}$ is a cover, giving the valid inequality $x_2 + x_3 + x_4 \leq 2$. We would however like to lift the cover inequality. What is the maximum value α for which

$$\alpha x_1 + x_2 + x_3 + x_4 \leq 2$$

is a valid inequality.

18.a) $\alpha = 3$

18.b) $\alpha = 2$

18.c) $\alpha = \frac{3}{2}$

18.d) $\alpha = 1$

18.e) $\alpha = \frac{1}{7}$

18.f) $\alpha = 0$

Minimizing the work-load of the raindeers (text question)

Santa Claus is planning the distribution of Christmas gifts. There are three types of gifts corresponding to how the children behaved during the year: A (very well), B (quite well) and C (not so well). For obvious reasons gifts of type A are the largest and most exciting, while type C are consolation prizes.

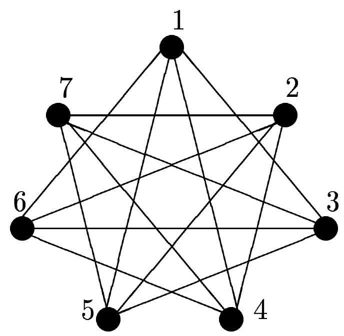
Exactly 1000 gifts should be brought out, but to avoid that too many children become envious, the amount of type A and C may differ by at most 100 gifts. If at least 500 gifts of type B are brought out, then at least 100 gifts of type A must be brought out also.

For bringing out all the gifts, one sleigh is needed pr. 100 gifts of type A, or one sleigh pr. 200 gifts of type B, or one sleigh pr. 300 gifts of type C. To avoid confusion during the hectic hours of Christmas Eve, no sleigh may contain more than one type of gifts. Santa Claus wishes to find a feasible solution using least possible sleighs.

Q 19 Formulate the problem as a Mixed Integer Programming model.

Covering the star (text question)

The vertex cover problem asks to choose a minimal number of nodes such that every edge is covered by at least one node.



Q 20 Let the variable $x_i \in \{0, 1\}$ indicate whether node i is selected. Formulate the problem as an integer programming model. For the graph shown above, derive the inequality

$$\sum_{i=1}^{7} x_i \geq 5$$

as a Chvatal-Gomory cut.