Friday, December 12

Program of the day:

- Delayed column generation — more formally
- Branch-and-price algorithms
- Example: VRP problems solved through delayed column generation
- “Evaluering, Spørgetime”

Dantzig-Wolfe Decomposition

The problem is split into a master problem and a subproblem

- Tighter bounds
- Better controle of subproblem
- Model may become (very) large

Delayed column generation

Write up the decomposed model gradually as needed

- Generate a few solutions to the subproblems
- Solve the master problem to LP-optimality
- Use the dual information to find most promising solutions to the subproblem
- Extend the master problem with the new subproblem solutions.

Decomposition

If model has “block” structure

$$
\begin{array}{llllllll}
\max & c^1x^1 & + & c^2x^2 & +\ldots+ & c^Kx^K & & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & +\ldots+ & A^Kx^K & = & b \\
 & D^1x^1 & + & & & & \leq & d_1 \\
 & & + & D^2x^2 & & & \leq & d_2 \\
 & & & & \ldots & & \leq & \vdots \\
 & & & & & D^Kx^K & \leq & d_K \\
 & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}
$$

Lagrangian relaxation

Objective becomes

$$
\begin{array}{l}
c^1x^1 + c^2x^2 + \ldots + c^Kx^K \\
-\lambda\left(A^1x^1 + A^2x^2 + \ldots + A^Kx^K - b\right)
\end{array}
$$

Decomposed into

$$
\begin{array}{llllll}
\max & c^1x^1 - \lambda A^1x^1 + c^2x^2 - \lambda A^2x^2 + \ldots + c^Kx^K - \lambda A^Kx^K + b & & & & \\
\text{s.t.} & D^1x^1 \quad + & & & & \leq d_1 \\
 & \quad + & D^2x^2 & & & \leq d_2 \\
 & & & \ldots & & \leq \vdots \\
 & & & & D^Kx^K & \leq d_K \\
 & x^1 \in \mathbb{Z}_+^{n_1} & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} &
\end{array}
$$

Model is separable

Dantzig-Wolfe decomposition

If model has “block” structure

$$
\begin{array}{lccccccl}
\max & c^1x^1 & + & c^2x^2 & +\ldots+ & c^Kx^K & & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & +\ldots+ & A^Kx^K & = & b \\
 & D^1x^1 & + & & & & \leq & d_1 \\
 & & + & D^2x^2 & & & \leq & d_2 \\
 & & & & \ldots & & \leq & \vdots \\
 & & & & & D^Kx^K & \leq & d_K \\
 & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}
$$

Substituting each set X^k, $k = 1,\ldots,K$ in original model getting *Master Problem*

$$\max c^1(\sum_{t\in T_1} \lambda_{1,t}x^{1,t}) + c^2(\sum_{t\in T_2} \lambda_{2,t}x^{2,t}) + \ldots + c^K(\sum_{t\in T_K} \lambda_{K,t}x^{K,t})$$

$$\text{s.t.}\quad A^1(\sum_{t\in T_1} \lambda_{1,t}x^{1,t}) + A^2(\sum_{t\in T_2} \lambda_{2,t}x^{2,t}) + \ldots + A^K(\sum_{t\in T_K} \lambda_{K,t}x^{K,t}) = b$$

$$\sum_{t\in T_k} \lambda_{k,t} = 1 \qquad k = 1,\ldots,K$$

$$\lambda_{k,t} \in \{0,1\}, \qquad t \in T_k \;\; k = 1,\ldots,K$$

Strength of linear master model

Solving LP-relaxation of master problem, is equivalent to (Wolsey Prop 11.1)

$$
\begin{array}{lccccccc}
\max & c^1x^1 & + & c^2x^2 & +\ldots+ & c^kx^k & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & +\ldots+ & A^kx^k & =b \\
& x^1 \in \operatorname{conv}(X^1) & & x^2 \in \operatorname{conv}(X^2) & \ldots & x^k \in \operatorname{conv}(X^k) &
\end{array}
$$

Strength of Lagrangian relaxation

- z^{LPM} be LP-solution value of master problem
- z^{LD} be solution value of lagrangian dual problem

(Theorem 11.2)

$$z^{LPM} = z^{LD}$$

Delayed column generation, linear master

(*minimization problem*)

Run Simplex algorithm as if complete master problem was known

- Start with a basis solution
- Solve
$$x_B = A_B^{-1} b$$
and find dual variables
$$y = c_B A_B^{-1}$$
- When choosing entering variable solve pricing problem which minimizes reduced costs
$$c_j^r = c_j - yA_j$$
- If $c_j^r < 0$ add corresponding column A_j to model and repeat
- If $c_j^r \geq 0$ stop

Cutting Stock Problem

(*minimization problem*)

a_{ij} is number of pieces of type i cut from pattern j
Master problem,

$$\begin{array}{ll} \min & \sum_{j=1}^{n} u_j \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} u_j \geq b_i \quad i = 1, \ldots, m, j = 1, \ldots, n \\ & x_{ij} \in \mathbb{Z}_+ \end{array}$$

Solving linear master through delayed column generation

- Start with patterns which only contain one type i
- Solve restricted master
- Dual variables y_i say how "attractive" a type i is
- Pricing problem

$$\begin{aligned} z^S = \min \quad & 1 - \sum_{i=1}^{m} y_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} w_i x_i \leq L \\ & x \geq 0, \text{integer} \end{aligned}$$

- stop if $z^S \geq 0$

Terminology

- Master Problem
- Restricted Master Problem
- Subproblem or Pricing Problem
- Branch and cut:
  Branch-and-bound algorithm using cuts to strengthen bounds.
- Branch and price:
  Branch-and-bound algorithm using column generation to derive bounds.
- One says that discarded columns are “priced out”.

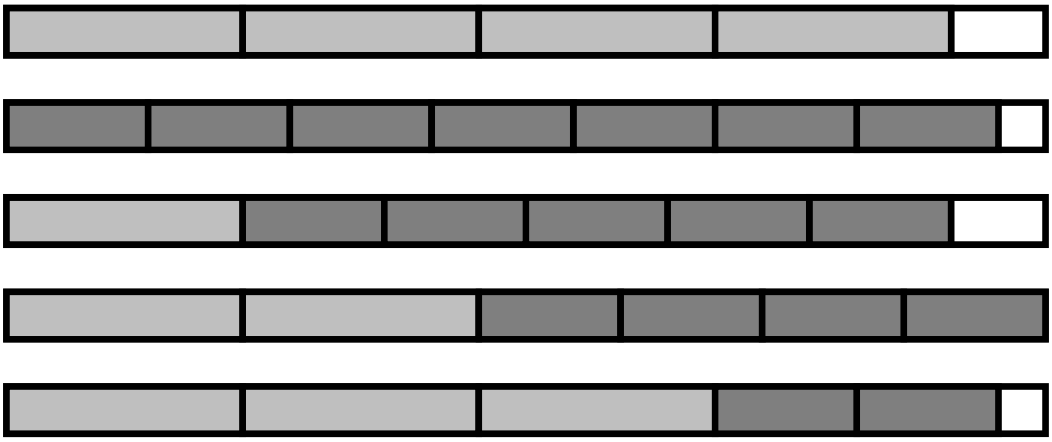
Branch-and-price

- LP-solution of master problem may have fractional solutions
- Branch-and-bound for getting IP-solution
- In each node solve LP-relaxation of master
- Subproblem may change when we add constraints to master problem
- Branching strategy should make subproblem easy to solve

Branch-and-price, example

The matrix A contains all different cutting patterns

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 7 & 5 & 4 & 2 \end{pmatrix}$$



Problem

$$\begin{aligned} \text{minimize} \quad & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \text{subject to} \quad & 4\lambda_1 + 0\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 \geq 7 \\ & 0\lambda_1 + 7\lambda_2 + 5\lambda_3 + 4\lambda_4 + 2\lambda_5 \geq 3 \\ & \lambda_j \in \mathbb{Z}_+ \end{aligned}$$

LP-solution $\lambda_1 = 1.375, \lambda_4 = 0.75$

Branch on $\lambda_1 = 0$, $\lambda_1 = 1$, $\lambda_1 = 2$

- Column generation may not generate pattern (4,0)
- Pricing problem is knapsack problem with pattern forbidden

Tailing off effect

Column generation may converge slowly in the end

- We do not need exact solution, just lower bound
- Solving master problem for subset of columns does not give valid lower bound (why?)
- Instead we may use Lagrangian relaxation of joint constraint
- “guess” lagrangian multipliers equal to dual variables from master problem

Heuristic solution

- Restricted master problem will only contain a subset of the columns
- We may solve restricted master problem to IP-optimality
- Restricted master is a “set-covering-like” problem which is not too difficult to solve

Vehicle Routing Problem with Time Windows

A delivery company shall visit 5 customers in some pre-described time-intervals $[a_i, b_i]$ as follows:

customer i	a_i	b_i
1	1	1
2	2	4
3	1	2
4	3	4
5	2	3

The depot is considered as node 0 and 6. The travel time t_{ij} between customer i and j, and corresponding cost c_{ij} is

$$c_{ij} = t_{ij} = |j - i| \quad i = 1, \dots, 6, \; j = 0, \dots, 6$$

There are at most five vehicles available, and the cost of a vehicle is 6 units, i.e. $c_{0j} = 6$. The travel time from the depot to the first customer is $t_{0j} = 1$.

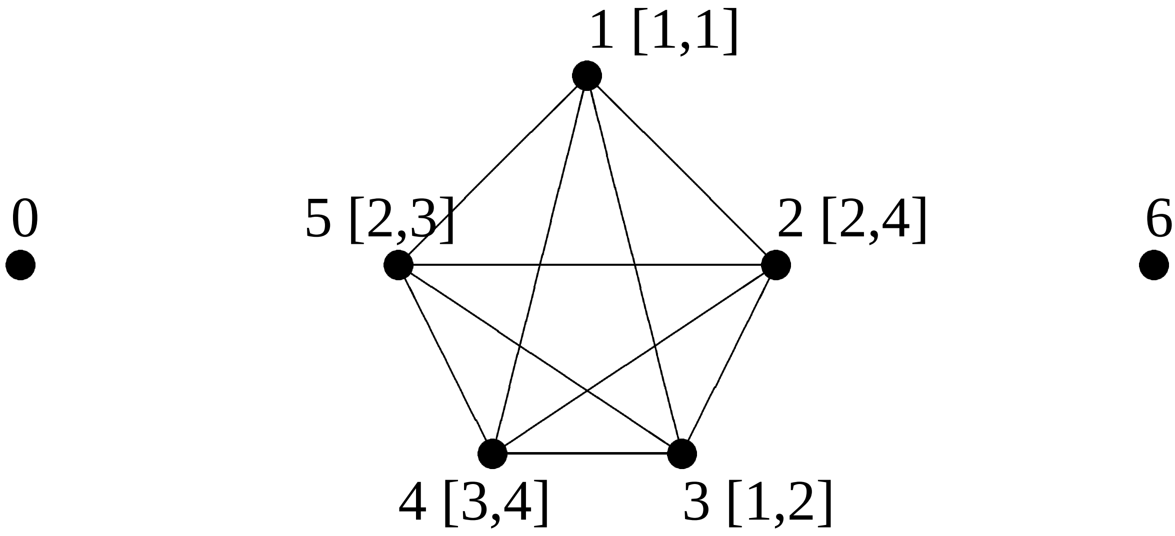
1 Find a feasible initial solution to the problem.

2 Use column generation to solve the problem to LP-optimality.

3 Write up all feasible routes, and solve the problem using CPLEX to integer optimality.

4 Compare the LP solution with the integer solution.

Graph



$$c_{ij} = t_{ij} = |j - i| \qquad c_{0j} = 6,\ t_{0j} = 1$$

Possible tours (in lexicographic order, dominated removed)

tour	cost
0,1,2,4,6	11
0,(wait),2,4,6	10
0,3,2,4,6	11
0,3,(wait),4,6	10
0,3,5,6	9
0,3,5,4,6	11
0,(wait),(wait),4,6	8
0,(wait),5,4,6	10
0,(wait),5,6	7

Column Generation

- **Initial Solution**. We can choose the trivial routes:

tour	cost
0,1,6	11
0,2,6	10
0,3,6	9
0,4,6	8
0,5,6	7

which gives a solution of 45.

- **Column generation**. We start with formulation

$$\begin{array}{lllllll} \min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & \\ \text{s.t.} & x_1 & & & & & \geq 1 \\ & & x_2 & & & & \geq 1 \\ & & & x_3 & & & \geq 1 \\ & & & & x_4 & & \geq 1 \\ & & & & & x_5 & \geq 1 \end{array}$$

the dual variables corresponding to the constraints are

$$y_1 = 11, \quad y_2 = 10, \quad y_3 = 9, \quad y_4 = 8, \quad y_5 = 7$$

The reduced cost of a tour is now the "original" cost of the edges minus the dual variables corresponding to the customers visited. The most beneficial tour is $(0,1,2,4,6)$ having the cost $11-(11+10+8) = -18$. Since the reduced cost of the new column is negative, we add it to the problem, getting:

$$\begin{array}{llllllll}
\min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & \\
\text{s.t.} & x_1 & & & & & +x_6 & \geq 1 \\
& & x_2 & & & & +x_6 & \geq 1 \\
& & & x_3 & & & & \geq 1 \\
& & & & x_4 & & +x_6 & \geq 1 \\
& & & & & x_5 & & \geq 1
\end{array}$$

the dual variables are

$$y_1 = 0, \quad y_2 = 3, \quad y_3 = 9, \quad y_4 = 8, \quad y_5 = 7$$

The most beneficial tour is $(0,3,5,4,6)$ having the reduced cost $11-(9+8+7) = -13$, which is added to the problem:

$$\begin{array}{lllllllll}
\min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & +11x_7 & \\
\text{s.t.} & x_1 & & & & & +x_6 & & \geq 1 \\
& & x_2 & & & & +x_6 & & \geq 1 \\
& & & x_3 & & & & +x_7 & \geq 1 \\
& & & & x_4 & & +x_6 & +x_7 & \geq 1 \\
& & & & & x_5 & & +x_7 & \geq 1
\end{array}$$

the dual variables are

$$y_1 = 1, \quad y_2 = 10, \quad y_3 = 4, \quad y_4 = 0, \quad y_5 = 7$$

Adding the tour $(0,3,2,4,6)$ at reduced cost $11-(10+4+0) = -3$ gives

$$\begin{array}{llllllllll}
\min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & +11x_7 & +11x_8 & \\
\text{s.t.} & x_1 & & & & & +x_6 & & & \geq 1 \\
& & x_2 & & & & +x_6 & & +x_8 & \geq 1 \\
& & & x_3 & & & & +x_7 & +x_8 & \geq 1 \\
& & & & x_4 & & +x_6 & +x_7 & +x_8 & \geq 1 \\
& & & & & x_5 & & +x_7 & & \geq 1
\end{array}$$

with dual variables

$$y_1 = 4, \quad y_2 = 7, \quad y_3 = 4, \quad y_4 = 0, \quad y_5 = 7$$

The most beneficial tour is $(0,3,5,6)$, having reduced cost $9-(4+7) = -2$. Thus

$$\begin{array}{llllllllllll}
\min & 11x_1 & +10x_2 & +9x_3 & +8x_4 & +7x_5 & +11x_6 & +11x_7 & +11x_8 & +9x_9 & \\
\text{s.t.} & x_1 & & & & & +x_6 & & & & \geq 1 \\
 & & x_2 & & & & +x_6 & & +x_8 & & \geq 1 \\
 & & & x_3 & & & & +x_7 & +x_8 & +x_9 & \geq 1 \\
 & & & & x_4 & & +x_6 & +x_7 & +x_8 & & \geq 1 \\
 & & & & & x_5 & & +x_7 & & +x_9 & \geq 1
\end{array}$$

with dual variables

$$y_1 = 2, \quad y_2 = 9, \quad y_3 = 2, \quad y_4 = 0, \quad y_5 = 7$$

Now, all tours have reduced costs ≥ 0, so we are done. The bound found by column generation is thus $z_{LP} = 20$.

- **Write up all feasible solutions**. The problem with all 9 tours added to the formulation, solved to IP-optimality, gives the solution $z_{IP} = 20$.
- **Compare bounds**. We have $z_{IP} = z_{LP}$ so the bounds are tight.