Wednesday, December 10

Program of the day:

- ”konkurrence P2”
- ”spørgetime, pensum”
- Dantzig-Wolfe reformulation of an IP
- Master problem
- Subproblem
- Linear Programming – canonical form, reduced costs, etc. (Taha chapter 7)
- Example: Cutting stock problem

Dantzig-Wolfe Decomposition

Motivation

- If you have a large or difficult problem: split it up into smaller pieces

Applications

- Facility location problems
- Cutting Stock problems
- Air-crew Scheduling
- Vehicle Routing Problems
- ...

Two currently most promising directions for MIP:

- Branch-and-price
- Branch-and-cut

Dantzig-Wolfe Decomposition

The problem is split into a master problem and a subproblem

- Tighter bounds
- Better controle of subproblem
- Model may become (very) large

Delayed column generation

Write up the decomposed model gradually as needed

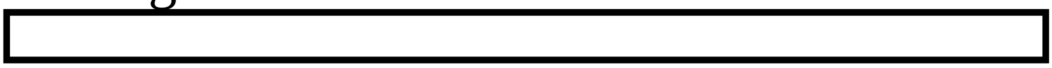
- Generate a few solutions to the subproblems
- Solve the master problem to LP-optimality
- Use the dual information to find most promising solutions to the subproblem
- Extend the master problem with the new subproblem solutions.

Motivation: Cutting stock problem

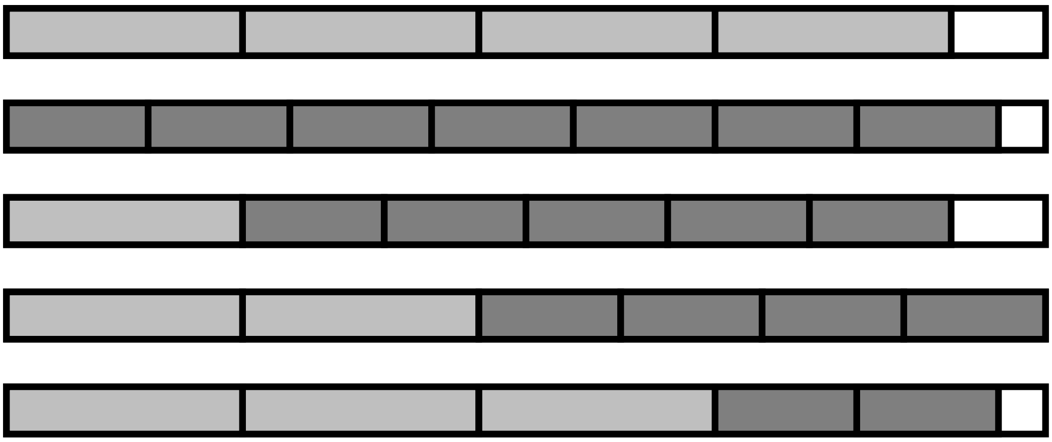
- Infinite number of raw stocks, having length L.
- Cut m piece types i, each having width w_i and demand b_i.
- Satisfy demands using least possible raw stocks.

Example:

- $w_1 = 5$, $b_1 = 7$ 
- $w_2 = 3$, $b_2 = 3$ 
- Raw length $L = 22$



Some possible cuts



Formulation 1

$$
\begin{array}{ll}
\text{minimize} & u_1 + u_2 + u_3 + u_4 + u_5 \\
\text{subject to} & 5x_{11} + 3x_{12} \leq 22u_1 \\
& 5x_{21} + 3x_{22} \leq 22u_2 \\
& 5x_{31} + 3x_{32} \leq 22u_3 \\
& 5x_{41} + 3x_{42} \leq 22u_4 \\
& 5x_{51} + 3x_{52} \leq 22u_5 \\
& x_{11} + x_{21} + x_{31} + x_{41} + x_{51} \geq 7 \\
& x_{12} + x_{22} + x_{32} + x_{42} + x_{52} \geq 3 \\
& u_j \in \{0, 1\} \\
& x_{ij} \in \mathbb{Z}_+
\end{array}
$$

LP-relaxation gives solution value $z = 2$ with

$$u_1 = u_2 = 1, x_{11} = 2.6, x_{12} = 3, x_{21} = 4.4$$

Block structure

min	u_1	$+u_2$	$+u_3$	$+u_4$	$+u_5$	
s.t	x_{11}	$+x_{21}$	$+x_{31}$	$+x_{41}$	$+x_{51}$	≥ 7
	x_{12}	$+x_{22}$	$+x_{32}$	$+x_{42}$	$+x_{51}$	≥ 3
	$5x_{11} + 3x_{12} - 22u_1$					≤ 0
		$5x_{21} + 3x_{22} - 22u_2$				≤ 0
			$5x_{31} + 3x_{32} - 22u_3$			≤ 0
				$5x_{41} + 3x_{42} - 22u_4$		≤ 0
					$5x_{51} + 3x_{52} - 22u_5$	≤ 0

Formulation 2

The matrix A contains all different cutting patterns
All (undominated) patterns:

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 7 & 5 & 4 & 2 \end{pmatrix}$$

Problem

$$\begin{array}{ll} \text{minimize} & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \text{subject to} & 4\lambda_1 + 0\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 \geq 7 \\ & 0\lambda_1 + 7\lambda_2 + 5\lambda_3 + 4\lambda_4 + 2\lambda_5 \geq 3 \\ & \lambda_j \in \mathbb{Z}_+ \end{array}$$

LP-relaxation gives solution value $z = 2.125$ with

$$\lambda_1 = 1.375, \lambda_4 = 0.75$$

Due to integer property a lower bound is $\lceil 2.125 \rceil = 3$.
Optimal solution value is $z^* = 3$.

Round up LP-solution getting heuristic solution $z_H = 3$.

Decomposition

If model has "block" structure

$$\begin{array}{llllllll} \max & c^1x^1 & + & c^2x^2 & +\ldots+ & c^Kx^K & & \\ \text{s.t.} & A^1x^1 & + & A^2x^2 & +\ldots+ & A^Kx^K & = & b \\ & D^1x^1 & + & & & & \leq & d_1 \\ & & + & D^2x^2 & & & \leq & d_2 \\ & & & & \ldots & & \leq & \vdots \\ & & & & & D^Kx^K & \leq & d_K \\ & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & & \end{array}$$

Lagrangian relaxation

Objective becomes

$$c^1x^1 + c^2x^2 + \ldots + c^Kx^K$$
$$-\lambda\left(A^1x^1 + A^2x^2 + \ldots + A^Kx^K - b\right)$$

Decomposed into

$$\begin{array}{llllll} \max & c^1x^1 - \lambda A^1x^1 + & c^2x^2 - \lambda A^2x^2 + & \ldots + & c^Kx^K - \lambda A^Kx^K + & b \\ \text{s.t.} & D^1x^1 \quad + & & & & \leq d_1 \\ & \quad\quad\quad + & D^2x^2 & & & \leq d_2 \\ & & & \ldots & & \leq \vdots \\ & & & & D^Kx^K & \leq d_K \\ & x^1 \in \mathbb{Z}_+^{n_1} & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & \end{array}$$

Model is separable

Dantzig-Wolfe decomposition

If model has "block" structure

$$\begin{array}{lccccccc}
\max & c^1x^1 & + & c^2x^2 & +\ldots+ & c^Kx^K & & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & +\ldots+ & A^Kx^K & = & b \\
 & D^1x^1 & + & & & & \leq & d_1 \\
 & & + & D^2x^2 & & & \leq & d_2 \\
 & & & & \ldots & & \leq & \vdots \\
 & & & & & D^Kx^K & \leq & d_K \\
 & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}$$

Describe each set X^k, $k = 1, \ldots, K$

$$\begin{array}{lcccccc}
\max & c^1x^1 & + & c^2x^2 & +\ldots+ & c^Kx^K & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & +\ldots+ & A^Kx^K & = b \\
 & x^1 \in X^1 & & x^2 \in X^2 & \ldots & x^K \in X^K &
\end{array}$$

where $X^k = \{x^k \in \mathbb{Z}_+^{n_k} : D^kx^k \leq d_k\}$

Assuming that X^k has finite number of points $\{x^{k,t}\}$ $t \in T_k$

$$X^k = \left\{ \begin{array}{ll} x^k \in \mathbb{R}^{n_k} : & x^k = \sum_{t \in T_k} \lambda_{k,t} x^{k,t}, \\ & \sum_{t \in T_k} \lambda_{k,t} = 1, \\ & \lambda_{k,t} \in \{0,1\}, t \in T_k \end{array} \right\}$$

Dantzig-Wolfe decomposition

Substituting X^k in original model getting *Master Problem*

$$\max c^1(\sum_{t\in T_1}\lambda_{1,t}x^{1,t}) + c^2(\sum_{t\in T_2}\lambda_{2,t}x^{2,t}) + \ldots + c^K(\sum_{t\in T_K}\lambda_{K,t}x^{K,t})$$

$$\text{s.t.}\quad A^1(\sum_{t\in T_1}\lambda_{1,t}x^{1,t}) + A^2(\sum_{t\in T_2}\lambda_{2,t}x^{2,t}) + \ldots + A^K(\sum_{t\in T_K}\lambda_{K,t}x^{K,t}) = b$$

$$\sum_{t\in T_k}\lambda_{k,t} = 1 \qquad k = 1,\ldots,K$$

$$\lambda_{k,t} \in \{0,1\}, \qquad t \in T_k \;\; k = 1,\ldots,K$$

Strength of linear master model

Solving LP-relaxation of master problem, is equivalent to (Wolsey Prop 11.1)

$$\begin{array}{llllllll} \max & c^1x^1 & + & c^2x^2 & +\ldots+ & c^kx^k & \\ \text{s.t.} & A^1x^1 & + & A^2x^2 & +\ldots+ & A^kx^k & =b \\ & x^1 \in \operatorname{conv}(X^1) & & x^2 \in \operatorname{conv}(X^2) & \ldots & x^k \in \operatorname{conv}(X^k) & \end{array}$$

Proof: Consider LP-relaxation

$$\max c^1(\sum_{t\in T_1} \lambda_{1,t}x^{1,t}) + c^2(\sum_{t\in T_2} \lambda_{2,t}x^{2,t}) + \ldots + c^K(\sum_{t\in T_K} \lambda_{K,t}x^{K,t})$$

$$\text{s.t.} \quad A^1(\sum_{t\in T_1} \lambda_{1,t}x^{1,t}) + A^2(\sum_{t\in T_2} \lambda_{2,t}x^{2,t}) + \ldots + A^K(\sum_{t\in T_K} \lambda_{K,t}x^{K,t}) = b$$

$$\sum_{t\in T_k} \lambda_{k,t} = 1 \qquad\qquad k = 1,\ldots,K$$

$$\lambda_{k,t} \geq 0, \qquad t \in T_k \qquad k = 1,\ldots,K$$

Informally speaking we have

- joint constraint is solved to LP-optimality
- block constraints are solved to IP-optimality

Strength of Lagrangian relaxation

- z^{LPM} be LP-solution value of master problem
- z^{LD} be solution value of lagrangian dual problem

(Theorem 11.2)

$$z^{LPM} = z^{LD}$$

Proof: Lagrangian relaxing joint constraint in

$$\begin{array}{llllllll} \max & c^1x^1 & + & c^2x^2 & + \ldots + & c^Kx^K & & \\ \text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^Kx^K & = & b \\ & D^1x^1 & + & & & & \leq & d_1 \\ & & + & D^2x^2 & & & \leq & d_2 \\ & & & & \ldots & & \leq & \vdots \\ & & & & & D^Kx^K & \leq & d_K \\ & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & & \end{array}$$

Using result next page

$$\begin{array}{llllllll} \max & c^1x^1 & + & c^2x^2 & + \ldots + & c^kx^k & & \\ \text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^kx^k & = & b \\ & x^1 \in \operatorname{conv}(X^1) & & x^2 \in \operatorname{conv}(X^2) & \ldots & x^k \in \operatorname{conv}(X^k) & & \end{array}$$

Strength of Lagrangian Relaxation (section 10.2)

Integer Programming Problem

$$\begin{array}{rl} \text{maximize} & cx \\ \text{subject to} & Ax \le b \\ & Dx \le d \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

Lagrange Relaxation, multipliers $\lambda \ge 0$

$$\begin{array}{rl} \text{maximize} & z_{LR}(\lambda) = cx - \lambda(Dx - d) \\ \text{subject to} & Ax \le b \\ & x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, n \end{array}$$

for best multiplier $\lambda \ge 0$

$$\boxed{\max \left\{ cx : Dx \le d, x \in \operatorname{conv}(Ax \le b, x \in \mathbb{Z}_+) \right\}}$$

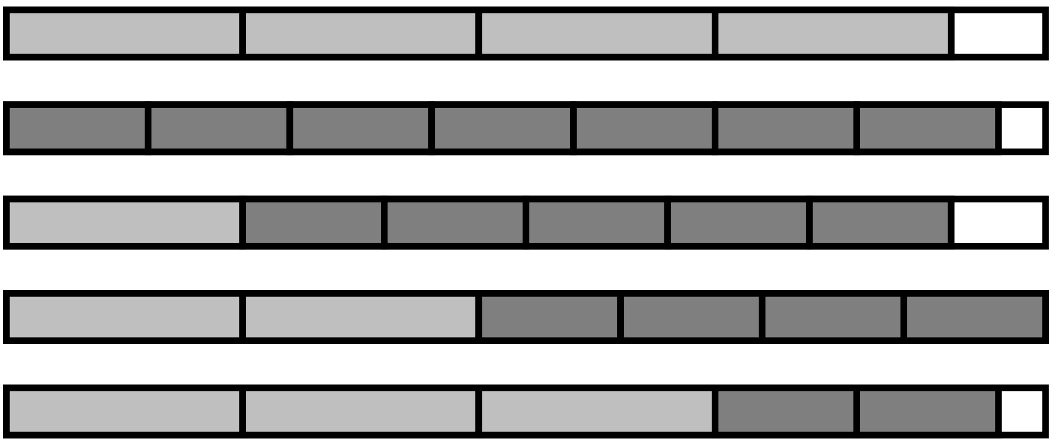
Delayed column generation, linear master

- Master problem can (and will) contain many columns
- To find bound, solve LP-relaxation of master
- Delayed column generation gradually writes up master

Delayed column generation, linear master

- $w_1 = 5$, $b_1 = 7$ 
- $w_2 = 3$, $b_2 = 3$ 
- Raw length $L = 22$

Some possible cuts



In matrix form

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 & \cdots \\ 0 & 7 & 5 & 4 & 2 & \cdots \end{pmatrix}$$

LP-problem

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

where

- $b = (7,3)$,
- $x = (x_1, x_2, x_3, x_4, x_5, \cdots)$
- $c = (1,1,1,1,1,\cdots)$.

Simplex in matrix form (Taha section 7.1)

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array}$$

Reformulation in matrix form

$$\begin{pmatrix} 1 & -c \\ 0 & A \end{pmatrix} \begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Assume

- x_B set of basis variables
- c_B coefficients in c corresponding to basis
- A_B coefficients in A corresponding to basis

Equivalent form

$$\begin{pmatrix} 1 & c_B A_B^{-1} A - c \\ 0 & A_B^{-1} A \end{pmatrix} \begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} c_B A_B^{-1} b \\ A_B^{-1} b \end{pmatrix}$$

Table page 295 in Taha

basis	x_j	solution
z	$c_B A_B^{-1} A_j - c_j$	$c_B A_B^{-1} b$
x_B	$A_B^{-1} A_j$	$A_B^{-1} b$

Simplex in matrix form

Basis variables and non-basis variables

$$x_B = (x_1, x_2) \qquad\qquad x_N = (x_3, x_4, x_5, \cdots)$$

split matrix

$$A_B = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix} \qquad\qquad A_N = \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 5 & 4 & 2 & \cdots \end{pmatrix}$$

reformulated problem

$$\begin{array}{ll} \text{min} & c_B x_B + c_N x_N \\ \text{s.t.} & A_B x_B + A_N x_N = b \\ & x \geq 0 \end{array}$$

Simplex algorithm sets $x_N = 0$ and solves $A_B x_B = b$ getting

$$x_B = A_B^{-1} b$$

corresponding objective function

$$z = c_B A_B^{-1} b + x_N (c_N - c_B A_B^{-1} A_N)$$

Since dual variables $y = c_B A_B^{-1}$ we have

$$z = yb + x_N (c_N - y A_N)$$

Delayed column generation (example)

- $w_1 = 5$, $b_1 = 7$ 
- $w_2 = 3$, $b_2 = 3$ 
- Raw length $L = 22$

Initially we choose only the trivial cutting patterns

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix}$$

Solve LP-problem

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

i.e.

$$\begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

with solution $x_1 = \frac{7}{4}$ and $x_2 = \frac{3}{7}$.
The dual variables are $y = c_B A_B^{-1}$ i.e.

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{7} \end{pmatrix}$$

Small example (continued)

Find entering variable

$$\begin{array}{lcl} A & = & \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 5 & 4 & 2 & \cdots \end{pmatrix} \qquad \begin{array}{l} \frac{1}{4} \leftarrow y_1 \\ \frac{1}{7} \leftarrow y_2 \end{array} \\ c_N - yA_N & = & \left(1 - \frac{27}{28} \;\; 1 - \frac{30}{28} \;\; 1 - \frac{29}{28} \;\; \cdots \right) \end{array}$$

We could also solve optimization problem

$$\begin{array}{ll} \min & 1 - \frac{1}{4}x_1 - \frac{1}{7}x_2 \\ \text{s.t.} & 5x_1 + 3x_2 \leq 22 \\ & x \geq 0, \text{integer} \end{array}$$

which is equivalent to knapsack problem

$$\begin{array}{ll} \max & \frac{1}{4}x_1 + \frac{1}{7}x_2 \\ \text{s.t.} & 5x_1 + 3x_2 \leq 22 \\ & x \geq 0, \text{integer} \end{array}$$

This problem has optimal solution $x_1 = 2$, $x_2 = 4$.
Reduced cost of entering variable

$$1 - 2\frac{1}{4} - 4\frac{1}{7} = 1 - \frac{30}{28} = -\frac{1}{14} < 0$$

Small example (continued)

Add new cutting pattern to A getting

$$A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 7 & 2 \end{pmatrix}$$

Solve problem to LP-optimality, getting primal solution

$$x_1 = \frac{5}{8}, x_3 = \frac{3}{2}$$

and dual variables

$$y_1 = \frac{1}{4}, y_2 = \frac{1}{8}$$

Note, we do not need to care about “leaving variable”
To find entering variable, solve

$$\begin{array}{ll} \max & \frac{1}{4}x_1 + \frac{1}{8}x_2 \\ \text{s.t.} & 5x_1 + 3x_2 \leq 22 \\ & x \geq 0, \text{integer} \end{array}$$

This problem has optimal solution $x_1 = 4$, $x_2 = 0$.
Reduced cost of entering variable

$$1 - 4\frac{1}{4} - 0\frac{1}{7} = 0$$

Terminate with $x_1 = \frac{5}{8}$, $x_3 = \frac{3}{2}$, and $z_{LP} = \frac{17}{8} = 2.125$.

Questions

- Will the process terminate?

Always improving objective value. Only a finite number of basis solutions.

- Can we repeat the same pattern?

No, since the objective functions is improved. We know the best solution among existing columns. If we generate an already existing column, then we will not improve the objective.

The Cutting Stock Problem (general model)

- Infinite number of raw stocks, having length L.
- Cut m piece types i, each having width w_i and demand b_i.
- Satisfy demands using least possible raw stocks.

IP-problem:

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} x_j & \\
\text{s.t.} \quad & \sum_{i=1}^{m} w_i a_{ij} \leq L x_j & j = 1, , n \\
& \sum_{j=1}^{n} a_{ij} \geq b_i, & i = 1, , m \\
& a_{ij} \geq 0, \text{integer} & i = 1, , m, j = 1, , n \\
& x_j \in \{0, 1\} & j = 1, , n
\end{aligned}
$$

where

- n upper bound on number of raw stocks.
- x_j is 1 if raw stock j is used.
- a_{ij} is number of pieces type i cut of stock j.

Cutting Stock: Better relaxation

- Write up all different cutting patterns
- Solve the LP-relaxation

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} x_j & \\
\text{s.t.} \quad & \sum_{i=1}^{m} w_i a_{ij} \leq L x_j & j = 1, \ldots, n \\
& \sum_{j=1}^{n} a_{ij} x_j \geq b_i, & i = 1, \ldots, m \\
& a_{ij} \geq 0, \text{integer} & i = 1, \ldots, m, j = 1, \ldots, n \\
& x_j \in \{0, 1\} & j = 1, , n
\end{aligned}
$$

Cutting Stock: Delayed column generation

Choosing variable to enter basis

$$z = yb + x_N(c_N - yA_N)$$

where $y = c_B A_B^{-1}$.
For every column a_j in A, the coefficient of variable x_j is

$$1 - \sum_{i=1}^{m} y_i a_{ij}$$

Thus to find the most negative coefficient we solve

$$\begin{aligned} z^S = \min \quad & 1 - \sum_{i=1}^{m} y_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} w_i x_i \leq L \\ & x \geq 0, \text{integer} \end{aligned}$$

Delayed column generation

- Columns in A are generated on the fly
- The process is greedy
- Terminate when no variable can enter basis ($z^S \geq 0$)
- Hopefully a small set of columns need to be generated