

November 26

Program of the day:

- Definition of facets, dimension
- Cover inequalities
- Separation of valid inequalities
- Lifting of inequalities
- Applications: The Traveling Salesman Problem, separation of subtour constraints

Deeper understanding of cuts, facets

- We would like to use the “best” formulation
- Dominance, redundancy, facets
- Facets are intuitively easy to understand
- How to prove that a valid inequality is facet defining?

Notice

$$\pi x \leq \pi_0$$

and

$$\lambda \pi x \leq \lambda \pi_0$$

are identical for any $\lambda > 0$

Dominance

$$\begin{aligned} & \text{maximize} && \dots \\ & \text{subject to} && 1x_1 + 3x_2 \leq 4 \\ & && 2x_1 + 4x_2 \leq 9 \\ & && x_1, x_2 \geq 0 \end{aligned}$$

Multiplying the second inequality with $u = \frac{1}{2}$

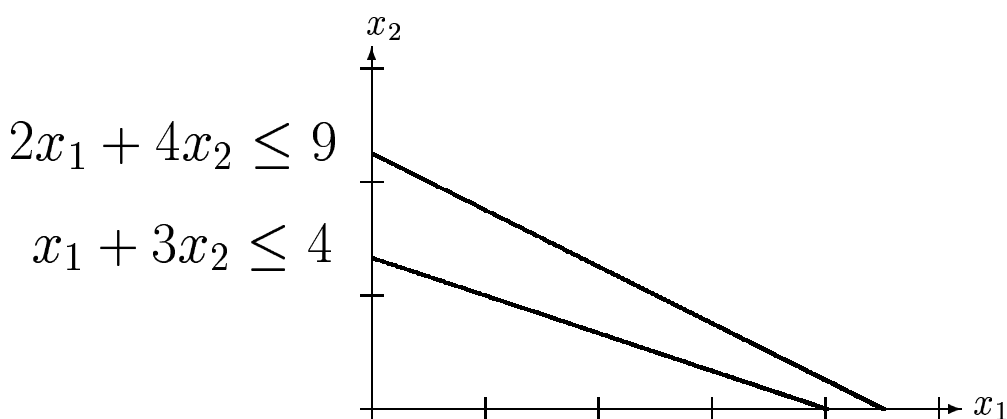
$$1x_1 + 2x_2 \leq \frac{9}{2}$$

First inequality dominates the second.

Dominance:

$$\pi x \leq \pi_0 \qquad \mu x \leq \mu_0$$

$\pi x \leq \pi_0$ dominates $\mu x \leq \mu_0$ if there exists $u > 0$ such that $\pi \geq u\mu$ and $\pi_0 \leq u\mu_0$.



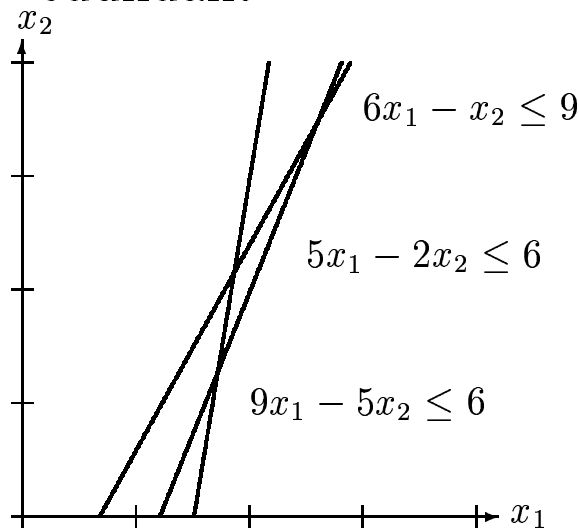
Redundance

$$\begin{aligned}
 & \text{maximize} && \dots \\
 & \text{subject to} && 6x_1 - x_2 \leq 9 \\
 & && 9x_1 - 5x_2 \leq 6 \\
 & && 5x_1 - 2x_2 \leq 6 \\
 & && x_1, x_2 \geq 0
 \end{aligned}$$

Multiplying the first two constraints with $u = (\frac{1}{3}, \frac{1}{3})$

$$5x_1 - 2x_2 \leq 5$$

Last inequality is redundant



Redundance:

$$\pi^i x \leq \pi_0^i, \quad i = 1, \dots, k$$

$$\mu x \leq \mu_0$$

Inequality $\mu x \leq \mu_0$ is redundant if there exists a vector $(u_1, \dots, u_k) \geq 0$ such that

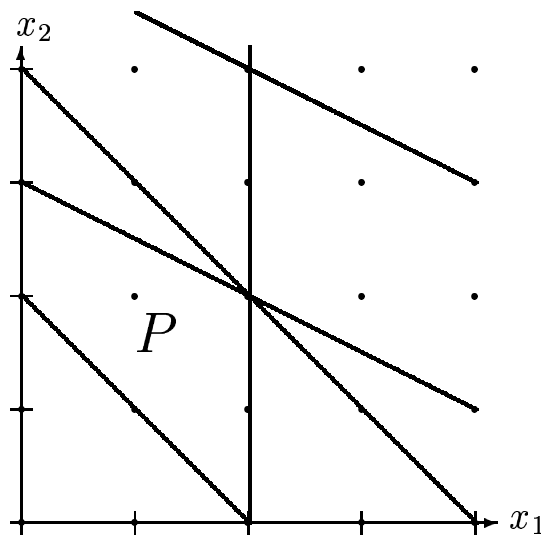
$$\left(\sum_{i=1}^k u_i \pi^i \right) x_i \leq \sum_{i=1}^k u_i \pi_0^i$$

dominates $\mu x \leq \mu_0$

Polyhedra, Facets

Polyhedra $P \subset \mathbb{R}^2$

$$\begin{aligned} \text{subject to } x_1 &\leq 2 \\ x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 10 \\ x_1 + 2x_2 &\leq 6 \\ x_1 + x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$



- $P \subset \mathbb{R}^2$ and “both directions are present”
- P is full-dimensional.
- The points $(2, 0)$, $(1, 1)$ and $(2, 2)$ are affinely independent points.
- The vectors $(2, 0, 1)$, $(1, 1, 1)$ and $(2, 2, 1)$ are linearly independent.
- The dimension of P is one less than the number of affinely independent points.

Polyhedra, Facets

Affinely independant

The points $x^1, x^2, \dots, x^k \in \mathbb{R}^n$ are affinely independant if the $k - 1$ directions $(x^2 - x^1), \dots, (x^k - x^1)$ are linearly independant.

Dimension

The dimension of P , denoted $\dim(P)$ is one less than the maximum number of affinely independant points in P .

A line is 1-dim

A plane is 2-dim

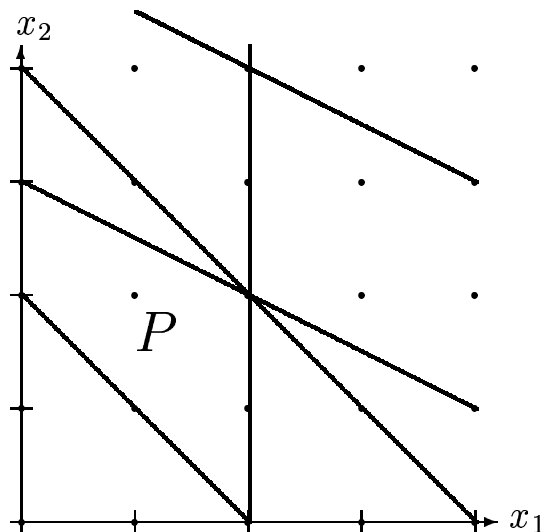
A box is 3-dim

Full-dimensional

The polyhedra $P \subseteq \mathbb{R}^n$ is full-dimensional if and only if $\dim(P) = n$.

Polyhedra, Facets

$$\begin{aligned}
 \text{subject to } x_1 &\leq 2 \\
 x_1 + x_2 &\leq 4 \\
 x_1 + 2x_2 &\leq 10 \\
 x_1 + 2x_2 &\leq 6 \\
 x_1 + x_2 &\geq 2 \\
 x_1, x_2 &\geq 0
 \end{aligned}$$



- $x_1 \leq 2$ defines a facet of P , as $(2, 0)$ and $(2, 2)$ are two affinely independent points in P .
- $x_1 + 2x_2 \leq 6$ defines a facet.
- $x_1 + x_2 \geq 2$ defines a facet.
- $x_1 \geq 0$ defines a facet.
- $x_1 + x_2 \leq 4$ is a face with one point $(2, 2) \in P$
- $x_1 + x_2 \leq 4$ is redundant: $u = (\frac{1}{2}, 0, 0, \frac{1}{2}, 0)$
- $x_2 \geq 0$ is redundant: $u = (1, 0, 0, 0, -1)$

Face, Facets

If $\pi x \leq \pi_0$ is a valid inequality of P then

$$F = \{x \in P : \pi x = \pi_0\}$$

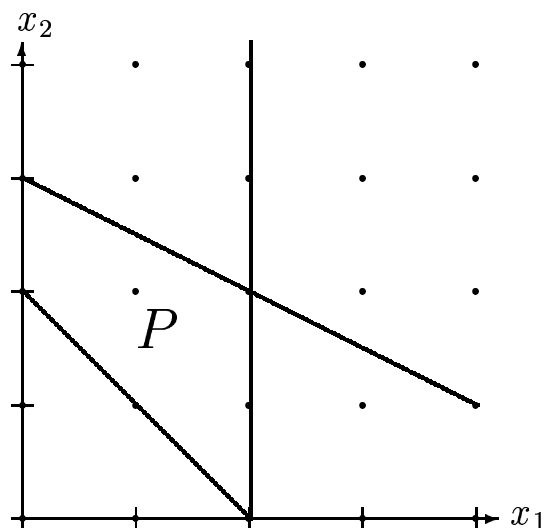
defines a face of P .

F is a facet of P iff

- F is a face of P
- $\dim(F) = \dim(P) - 1$

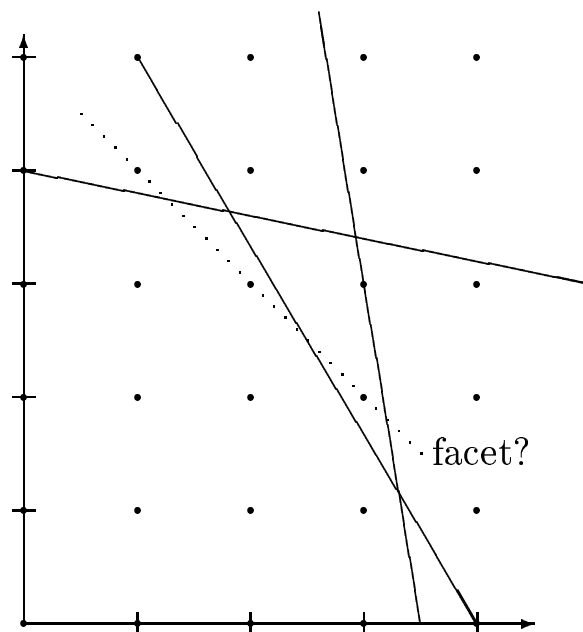
Minimal description of previous example

$$\begin{array}{rcl} \text{subject to } x_1 & & \leq 2 \\ & x_1 + 2x_2 & \leq 6 \\ & x_1 + x_2 & \geq 2 \\ & x_1 & \geq 0 \end{array}$$



IP-problems

$$P = \{Ax \leq b\} \cap \mathbb{Z}^2$$



- Dimension of P is 2
- The facet defining inequality must be valid
- A facet should have dimension 1
- There should be 2 affine independent points on a facet

Cover inequalities

$$11x_1 + 6x_2 - 6x'_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 13$$
$$x \in \{0, 1\}$$

To get positive coefficients we substitute $x_3 = 1 - x'_3$

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

Observe

- At most two of x_1 , x_2 and x_3 can be 1.
- At most two of x_1 , x_2 and x_6 can be 1.
- At most two of x_1 , x_5 and x_6 can be 1.
- At most three of x_3 , x_4 , x_5 and x_6 can be 1.

Cover inequalities

Consider the set

$$X = \left\{ x \in \mathbb{B}^n : \sum_{j=1}^n a_j x_j \leq b \right\}$$

We assume that $a_j \geq 0$ and $b \geq 0$.

Cover

A set $C \subseteq N$ is a cover if

$$\sum_{j \in C} a_j > b$$

A set $C \subseteq N$ is a minimal cover if $C \setminus \{j\}$ is not a cover for any $j \in C$

Cover Inequality

If C is a cover the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid for X .

Cover inequalities

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

Minimal cover inequalities

$$\begin{aligned}x_1 + x_2 + x_3 &\leq 2 \\x_1 + x_2 + x_6 &\leq 2 \\x_1 + x_5 + x_6 &\leq 2 \\x_3 + x_4 + x_5 + x_6 &\leq 3\end{aligned}$$

Extended cover inequalities for $C = \{3, 4, 5, 6\}$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

which dominates

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

Extended cover inequalities

If C is a cover for X , the extended cover inequality

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is valid, where

$$E(C) = C \cup \{j \in N : a_j \geq a_i \text{ for all } i \in C\}$$

Lifting Cover Inequalities

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \quad (1)$$

Have found cover inequality

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

What is the value of α_1 such that

$$\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3 \quad (2)$$

is valid for all $x \in X$?

Keeping $x_2 = x_7 = 0$ constraint (2) must be valid whenever (1) is valid.

- $x_1 = 0$ then (2) is valid.
- $x_1 = 1$ then we demand that

$$\alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

whenever

$$11 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19$$

Maximize $x_3 + x_4 + x_5 + x_6$ subject to the second inequality

$$\begin{aligned} \gamma = \text{maximize } & x_3 + x_4 + x_5 + x_6 \\ \text{subject to } & 11 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 \\ & x_i \in \{0, 1\} \end{aligned}$$

a Knapsack Problem with solution $\gamma = 1$.

Thus $\alpha_1 = 3 - \gamma = 2$.

Separation problem

The separation problem decides whether a LP-solution vector satisfies all constraints of a given family \mathcal{F} . If it does not, it must return a violated constraint in \mathcal{F}

Separation of cover inequalities

We consider a large IP model

$$\begin{aligned} & \text{maximize } cx \\ & \text{subject to } Ax \leq b \\ & \quad x \in \{0, 1\} \end{aligned}$$

Current solution $x = x'$ is fractional.

Pick a constraint

$$\sum_{i \in I} a_i x_i \leq b$$

Solve problem

$$\begin{aligned} \gamma = & \text{minimize } \sum_{i \in I} (1 - x'_i) \delta_i \\ & \text{subject to } \sum_{i \in I} a_i \delta_i \geq b + 1 \\ & \quad \delta_i \in \{0, 1\}, \quad i \in I. \end{aligned}$$

If $\gamma < 1$, let

$$C = \{i \in I : \delta_i = 1\}$$

New inequality

$$\sum_{i \in C} x_i \leq |C| - 1$$

C is a minimal cover

exercise

The inequality separates current $\delta = \delta'$

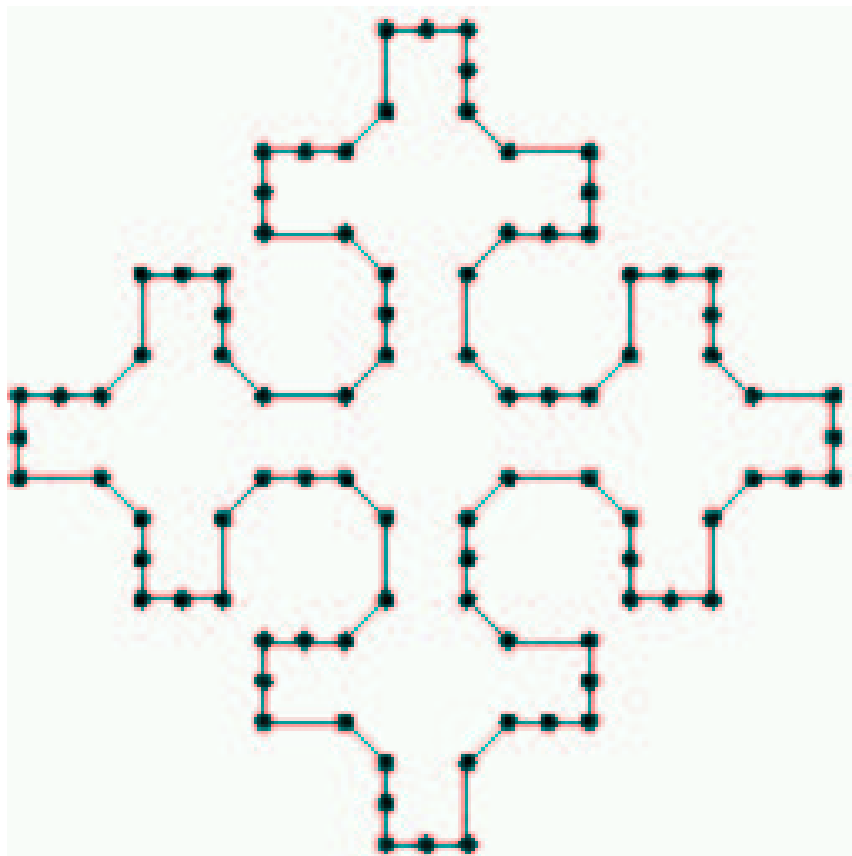
exercise

Traveling Salesman Problem

One of most famous and most applicable optimization problems

Given a finite number of "cities" along with the cost of travel between each pair of them, find the cheapest way of visiting all the cities and returning to your starting point.

Recently Applegate, Bixby, Chvatal, Cook solved USA13509 problem



Traveling Salesman Problem

- Set of V cities
- To each edge $e \in E$ is associated a cost c_e
- Visit each city exactly once
- Minimize travel cost
- $x_e = 1$ if edge e is used

Model 1

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(j)} x_e = 2 \quad , \quad j \in V \\ & \sum_{e \in E(S)} x_e \leq |S| - 1 \quad , \quad S \subset V, S \neq V \\ & x_e \in \{0, 1\} \end{aligned}$$

Model 2

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(j)} x_e = 2 \quad , \quad j \in V \\ & \sum_{e \in \delta(S)} x_e \geq 2 \quad , \quad S \subset V, S \neq V \\ & x_e \in \{0, 1\} \end{aligned}$$

degree constraint, subtour elimination constraint

Traveling Salesman Problem

Subtour LP

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(j)} x_e = 2 \quad , \quad j \in V \\ & \sum_{e \in \delta(S)} x_e \geq 2 \quad , \quad S \subset V, S \neq V \\ & 0 \leq x_e \leq 1 \end{aligned}$$

exponentially many constraints

Cutting plane algorithm:

- 1 solve problem without subtour elimination constraints getting x_e^*
- 2 if x_e^* is a hamilton cycle, stop
- 3 solve separation problem obtaining a valid inequality

$$\sum_{e \in \delta(S)} x_e \geq 2$$

such that

$$\sum_{e \in \delta(S)} x_e^* < 2$$

- 4 add the valid inequality to the problem and repeat

Traveling Salesman Problem

Separation problem:

- capacitated network (V, E, d)
- $d_e = x_e^*$
- find min cut in graph
- optimal solution has value less than 2 iff violated constraint exists
- min-cut can be found in $O(nm \log n)$ time where $n = |V|$ and $m = |E|$.
- try each pair of nodes, i.e. run min-cut $n(n - 1)/2$ times

$$c_e = \begin{pmatrix} - & 4 & 3 & 3 & 5 & 2 & 5 \\ - & - & 5 & 3 & 3 & 4 & 7 \\ - & - & - & 4 & 6 & 0 & 4 \\ - & - & - & - & 4 & 4 & 6 \\ - & - & - & - & - & 5 & 8 \\ - & - & - & - & - & - & 3 \\ - & - & - & - & - & - & - \end{pmatrix}$$

Traveling Salesman Problem

```

minimize
+ 4 x12 + 3 x13 + 3 x14 + 5 x15 + 2 x16 + 5 x17
      + 5 x23 + 3 x24 + 3 x25 + 4 x26 + 7 x27
            + 4 x34 + 6 x35 + 0 x36 + 4 x37
                  + 4 x45 + 4 x46 + 6 x47
                          + 5 x56 + 8 x57
                                  + 3 x67

```

subject to

```

      x12 + x13 + x14 + x15 + x16 + x17 = 2
x12 +      x23 + x24 + x25 + x26 + x27 = 2
x13 + x23 +      x34 + x35 + x36 + x37 = 2
x14 + x24 + x34 +      x45 + x46 + x47 = 2
x15 + x25 + x35 + x45 +      x56 + x57 = 2
x16 + x26 + x36 + x46 + x56 +      x67 = 2
x17 + x27 + x37 + x47 + x57 + x67 = 2

```

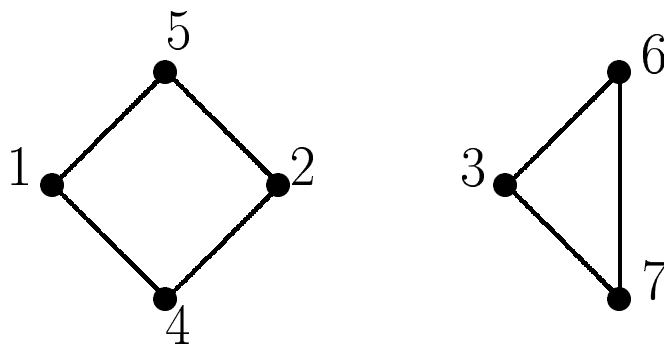
binary

```

      x12 x13 x14 x15 x16 x17
      x23 x24 x25 x26 x27
      x34 x35 x36 x37
      x45 x46 x47
      x56 x57
      x67

```

end



Traveling Salesman Problem

```

minimize
+ 4 x12 + 3 x13 + 3 x14 + 5 x15 + 2 x16 + 5 x17
      + 5 x23 + 3 x24 + 3 x25 + 4 x26 + 7 x27
            + 4 x34 + 6 x35 + 0 x36 + 4 x37
                  + 4 x45 + 4 x46 + 6 x47
                          + 5 x56 + 8 x57
                                  + 3 x67

```

```

subject to
      x12 + x13 + x14 + x15 + x16 + x17 = 2
x12 +      x23 + x24 + x25 + x26 + x27 = 2
x13 + x23 +      x34 + x35 + x36 + x37 = 2
x14 + x24 + x34 +      x45 + x46 + x47 = 2
x15 + x25 + x35 + x45 +      x56 + x57 = 2
x16 + x26 + x36 + x46 + x56 +      x67 = 2
x17 + x27 + x37 + x47 + x57 + x67 = 2

```

```

x13 + x23 + x34 + x35 +
x16 + x26 + x46 + x56 +
x17 + x27 + x47 + x57 >= 2

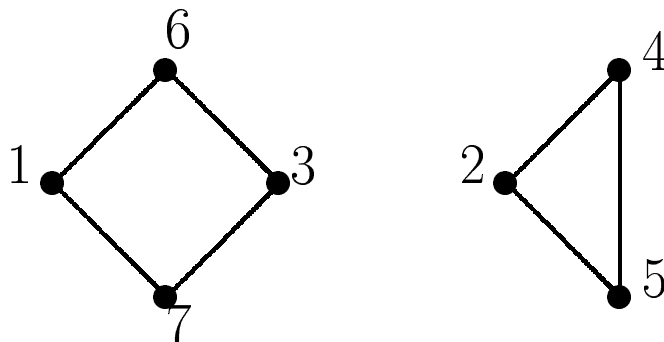
```

```

binary
      x12 x13 x14 x15 x16 x17
        x23 x24 x25 x26 x27
          x34 x35 x36 x37
            x45 x46 x47
              x56 x57
                x67

```

end



Traveling Salesman Problem

```

minimize
+ 4 x12 + 3 x13 + 3 x14 + 5 x15 + 2 x16 + 5 x17
      + 5 x23 + 3 x24 + 3 x25 + 4 x26 + 7 x27
            + 4 x34 + 6 x35 + 0 x36 + 4 x37
                  + 4 x45 + 4 x46 + 6 x47
                          + 5 x56 + 8 x57
                                  + 3 x67

```

```

subject to
      x12 + x13 + x14 + x15 + x16 + x17 = 2
x12 +      x23 + x24 + x25 + x26 + x27 = 2
x13 + x23 +      x34 + x35 + x36 + x37 = 2
x14 + x24 + x34 +      x45 + x46 + x47 = 2
x15 + x25 + x35 + x45 +      x56 + x57 = 2
x16 + x26 + x36 + x46 + x56 +      x67 = 2
x17 + x27 + x37 + x47 + x57 + x67 = 2

```

```

x13 + x23 + x34 + x35 +
x16 + x26 + x46 + x56 +
x17 + x27 + x47 + x57 >= 2

```

```

x12 + x23 + x26 + x27 +
x14 + x34 + x46 + x67 +
x15 + x35 + x56 + x57 >= 2

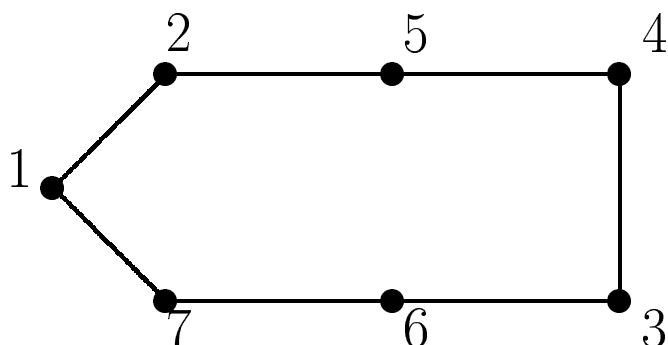
```

```

binary
      x12 x13 x14 x15 x16 x17
      x23 x24 x25 x26 x27
            x34 x35 x36 x37
                  x45 x46 x47
                          x56 x57
                                  x67

```

end



Prize Collecting Traveling Salesman Problem

- Set of N cities.
- Salesman starts in city 1.
- To each edge e is associated a cost c_e
- To each node j is associated a profit f_j
- Visit *at least* two other cities
- Maximize profit – cost.

Introduce variables

- $x_e = 1$ if edge e is used.
- $y_j = 1$ if node j is visited.

Formulation

$$\begin{aligned} \max \quad & \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(j)} x_e = 2y_j \quad , \quad j \in N \\ & \sum_{e \in E(S)} x_e \leq \sum_{i \in S \setminus \{k\}} y_i \quad , \quad k \in S, S \subseteq N \setminus \{1\} \\ & y_1 = 1 \\ & x_e \in \{0, 1\}, y_j \in \{0, 1\} \end{aligned}$$

Separation for generalized subtour constraints

Assume that we solve the ILP-problem

$$\begin{aligned}
 \max \quad & \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e \\
 \text{s.t.} \quad & \sum_{e \in \delta(j)} x_e = 2y_j, \quad j \in N \\
 & y_1 = 1 \\
 & x \in \{0, 1\}, y \in \{0, 1\}
 \end{aligned} \tag{3}$$

getting a solution (x^*, y^*) . How do we find a violated GSE constraint?

- $N' = N \setminus 1$
- $E' = E \setminus \{\delta(1)\}$
- $z_i = 1$ iff $i \in S$

A constraint for (k, S) is violated if

$$\sum_{e \in E'(S)} x_e^* > \sum_{i \in S \setminus \{k\}} y_i^*$$

This can be formulated as a maximization problem

$$\begin{aligned}
 \gamma = \max \quad & \sum_{e \in E'} x_e^* z_i z_j - \sum_{i \in N' \setminus \{k\}} y_i^* z_i \\
 \text{s.t.} \quad & z_k = 1 \\
 & z \in \{0, 1\}
 \end{aligned}$$

Separation for generalized subtour constraints

The quadratic 0-1 program

$$\begin{aligned} \gamma = \max \quad & \sum_{e=(i,j) \in E'} x_e^* z_i z_j - \sum_{i \in N' \setminus \{k\}} y_i^* z_i \\ \text{s.t.} \quad & z_k = 1 \\ & z \in \{0, 1\} \end{aligned}$$

can be reformulated using

$$w_{(i,j)} = 1 \Leftrightarrow z_i = 1 \text{ and } z_j = 1$$

but since we maximize only

$$w_{(i,j)} = 1 \Rightarrow z_i = 1 \text{ and } z_j = 1$$

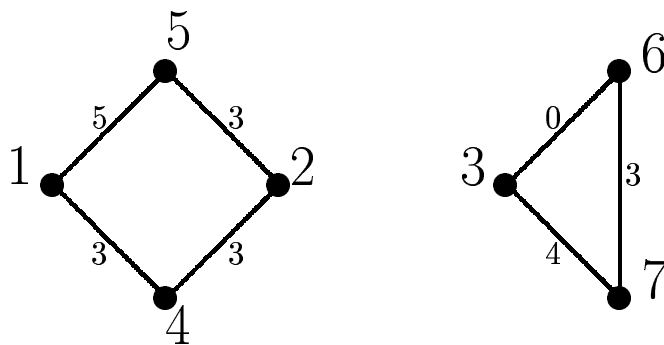
is needed

$$\begin{aligned} \gamma = \max \quad & \sum_{e=(i,j) \in E'} x_e^* w_e - \sum_{i \in N' \setminus \{k\}} y_i^* z_i \\ \text{s.t.} \quad & w_{(i,j)} \leq z_i, \quad (i,j) \in E' \\ & w_{(i,j)} \leq z_j, \quad (i,j) \in E' \\ & z_k = 1 \\ & w \in \{0, 1\}, z \in \{0, 1\} \end{aligned}$$

This formulation is TU and thus can be solved in polynomial time

Separation for generalized subtour constraints

$$f = (2, 4, 1, 3, 7, 1, 7) \text{ and } c_e = \begin{pmatrix} - & 4 & 3 & 3 & 5 & 2 & 5 \\ - & - & 5 & 3 & 3 & 4 & 7 \\ - & - & - & 4 & 6 & 0 & 4 \\ - & - & - & - & 4 & 4 & 6 \\ - & - & - & - & - & 5 & 8 \\ - & - & - & - & - & - & 3 \\ - & - & - & - & - & - & - \end{pmatrix}$$



The LP-relaxation of (3) gives the routes

$$(1, 5, 2, 4) \text{ and } (3, 6, 7)$$

The separation algorithm returns

$$x_{36} + x_{37} + x_{67} \leq y_3 + y_7$$

which cuts off the subtour (3, 6, 7).