

November 22

Program of the day:

- Cutting planes — a method to obtain tighter bounds and faster convergence to integer solutions (Wolsey chap. 8)
- Application: branch-and-cut algorithms

Introduction

- branch-and-bound: divide and conquer.
- cutting plane: add inequalities which separate fractional solution from solution space.

Development

- 50's cutting plane (Gomory: simplex, no \mathcal{NP} -hardness)
- 70's tighten formulation in preprocessing
- 80-90's branch-and-cut (Padberg, Rinaldi)

Preprocessing \rightarrow part of solution process

Definitions

- cuts: valid inequalities
- facets: inequalities defining convex hull

Cuts and facets are redundant for IP formulation
Tighten formulation for LP relaxation

Examples from last lesson

Preprocessing, integer variables

$$\begin{array}{ll} \text{maximize} & \dots \\ \text{subject to} & 7x_1 + 3x_2 - 4x_3 - 2x_4 \leq 1 \\ & -2x_1 + 7x_2 + 3x_3 + 4x_4 \leq 6 \\ & \quad - 2x_2 - 3x_3 - 6x_4 \leq -5 \\ & 3x_1 \quad \quad - 2x_3 \geq -1 \\ & x \in \mathbb{B}^4 \end{array}$$

Generating logical inequalities

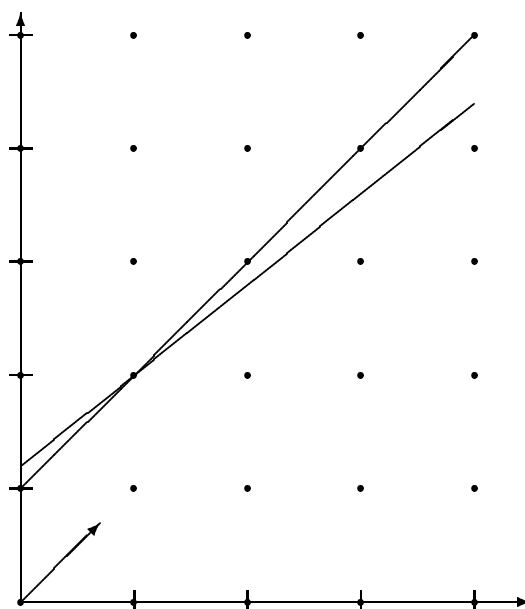
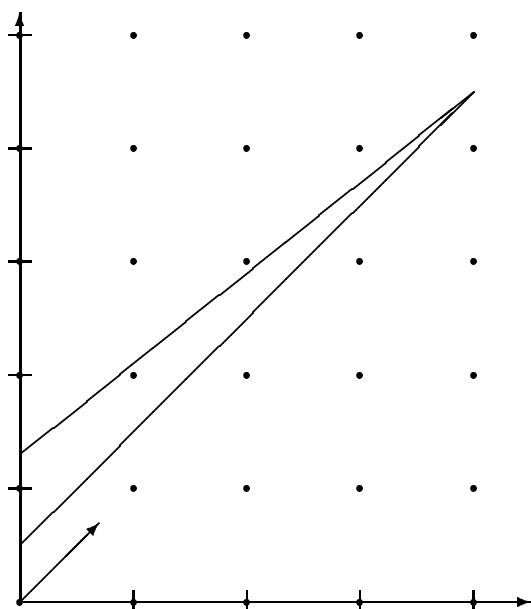
From constraint 1 we see that

- if $x_1 = 1$ and $x_2 = 1$ then infeasible, thus

$$x_1 + x_2 \leq 1$$

Examples from last lesson

$$\begin{aligned}
 &\text{maximize} && x_1 + x_2 \\
 &\text{subject to} && -2x_1 + 2x_2 \geq 1 \\
 &&& -8x_1 + 10x_2 \leq 13 \\
 &&& x_1, x_2 \geq 0, \text{ integer}
 \end{aligned}$$



Tightening formulation

$$\begin{array}{ll}
 -2x_1 + 2x_2 \geq 1 & -8x_1 + 10x_2 \leq 13 \\
 -x_1 + x_2 \geq 1/2 & -4x_1 + 5x_2 \leq 13/2 \\
 -x_1 + x_2 \geq 1 & -4x_1 + 5x_2 \leq 6
 \end{array}$$

Motivation

Integer programming problem (IP)

$$\max\{cx : x \in X\}$$

where $X = \{x : Ax \leq b, x \in \mathbb{Z}_+^n\}$. Reformulate to

$$\max\{cx : x \in \text{conv}(X)\}$$

For any c , an optimal solution to LP is also optimal to IP

Valid inequalities (def. 8.1)

Consider the problem:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

An inequality

$$\pi x \leq \pi_0$$

is a *valid inequality* for $X \subseteq \mathbb{R}^n$ if

$$\pi x \leq \pi_0 \text{ for all } x \in X$$

Characterisation of valid inequalities (sec. 8.3.2)

Consider the problem:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

where

$$X = \{y \in \mathbb{Z} : y \leq b\}$$

then the inequality

$$y \leq \lfloor b \rfloor$$

is valid for X

- Simple observation
- Complete characterisation

Overview of cuts

- Chvatal cuts
- Gomory cuts (Modular cuts)
- Chvatal-Gomory cuts
- Disjunctive cuts
- Cover inequalities
- Clique inequalities
- Problem specific cuts

Notice

- Cuts and facets are independant of objective function
- A tight formulation can be used for any objective

Example of Facets

The problem

$$\begin{array}{ll} \text{minimize} & 2x_1 + 7x_2 + 2x_3 \\ \text{subject to} & x_1 + 4x_2 + x_3 \geq 10 \\ & 4x_1 + 2x_2 + 2x_3 \geq 13 \\ & x_1 + x_2 - x_3 \geq 0 \\ & x_1, x_2, x_3 \geq 0, \text{ integer} \end{array}$$

has the facets

$$\begin{array}{ll} x_1 + 4x_2 + x_3 & \geq 10 \\ 2x_1 + x_2 + x_3 & \geq 7 \\ x_1 + x_2 - x_3 & \geq 0 \\ x_1 + 3x_2 + x_3 & \geq 9 \\ 2x_1 + 4x_2 + x_3 & \geq 13 \\ x_1 + x_2 + x_3 & \geq 5 \\ x_1 + 2x_2 & \geq 5 \\ 2x_1 + x_2 & \geq 4 \\ x_1 & \geq 0, \text{ integer} \\ x_2 & \geq 0, \text{ integer} \\ x_3 & \geq 0, \text{ integer} \end{array}$$

Using the new formulation we obtain an integer optimal solution by solving the LP-relaxed problem. (For any objective function).

Chvátal Cuts

Valid inequalities for a pure IP-model (minimization)

- 1 Add constraints, using suitable multipliers
- 2 Divide through by a common coefficient factor
- 3 Round up right-hand-side to the next integer

Example

$$\begin{array}{ll} \text{minimize} & 2x_1 + 7x_2 + 2x_3 \\ \text{subject to} & x_1 + 4x_2 + x_3 \geq 10 \quad (1) \\ & 4x_1 + 2x_2 + 2x_3 \geq 13 \quad (2) \\ & x_1 + x_2 - x_3 \geq 0 \quad (3) \\ & x_1 \geq 0 \quad (4) \\ & x_2 \geq 0 \quad (5) \\ & x_3 \geq 0 \quad (6) \\ & x_1, x_2, x_3 \text{ integer} \end{array}$$

1 times (2) is

$$4x_1 + 2x_2 + 2x_3 \geq 13$$

divide by two

$$2x_1 + x_2 + x_3 \geq 6\frac{1}{2}$$

left hand side is integral, thus round up right-hand

$$2x_1 + x_2 + x_3 \geq 7$$

Example (continued)

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 7x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + 4x_2 + x_3 \geq 10 & (1) \\ & 4x_1 + 2x_2 + 2x_3 \geq 13 & (2) \\ & x_1 + x_2 - x_3 \geq 0 & (3) \\ & x_1 \geq 0 & (4) \\ & x_2 \geq 0 & (5) \\ & x_3 \geq 0 & (6) \\ & x_1, x_2, x_3 \text{ integer} \end{aligned}$$

Facets

$$\begin{aligned} x_1 + 4x_2 + x_3 &\geq 10 & (a) \\ 2x_1 + x_2 + x_3 &\geq 7 & (b) \\ x_1 + x_2 - x_3 &\geq 0 & (c) \\ x_1 + 3x_2 + x_3 &\geq 9 & (d) \\ 2x_1 + 4x_2 + x_3 &\geq 13 & (e) \\ x_1 + x_2 + x_3 &\geq 5 & (f) \\ x_1 + 2x_2 &\geq 5 & (g) \\ 2x_1 + x_2 &\geq 4 & (h) \\ x_1, x_2, x_3 &\geq 0, \text{ integer} \end{aligned}$$

Obtained as

$$\begin{aligned} (d) & : 5 \times (1), 1 \times (b), 1 \times (6), \text{ divide } 7 \\ (f) & : 1 \times (1), 3 \times (b), 3 \times (6), \text{ divide } 7 \\ (g) & : 4 \times (1), 1 \times (b), 5 \times (3), \text{ divide } 11 \\ (h) & : 1 \times (b), 1 \times (6), 1 \times (4), \text{ divide } 2 \\ (e) & : 3 \times (d), 1 \times (b), 3 \times (g), \text{ divide } 4 \end{aligned}$$

Chvatal-Gomory cuts

$$\begin{aligned} & \text{maximize} && \sum_{j=1}^n c_j x_j \\ & \text{subject to} && \sum_{j=1}^n a_{1j} x_j \leq b_1 \\ & && \vdots \\ & && \sum_{j=1}^n a_{mj} x_j \leq b_m \\ & && x_j \in \mathbb{Z}_+, \quad j = 1, \dots, n \end{aligned}$$

Three-step procedure

- 1 Take a linear combination of the constraints

$$\sum_{j=1}^n \left(\sum_{i=1}^m u_i a_{ij} \right) x_j \leq \left(\sum_{i=1}^m u_i b_i \right)$$

in short

$$\sum_{j=1}^n a'_j x_j \leq b'$$

- 2 Since $x \geq 0$ implies $\sum_{j=1}^n (a'_j - \lfloor a'_j \rfloor) x_j \geq 0$ we have

$$\sum_{j=1}^n \lfloor a'_j \rfloor x_j \leq b'$$

- 3 Since $x_j \in \mathbb{Z}_+$ implies $\lfloor a'_j \rfloor x_j \in \mathbb{Z}$ we get

$$\sum_{j=1}^n \lfloor a'_j \rfloor x_j \leq \lfloor b' \rfloor$$

Chvatal-Gomory (Theorem 8.4)

$$X = \{x : Ax \leq b, x \in \mathbb{Z}_+^n\}$$

Every valid inequality for X can be obtained by applying the Chvatal-Gomory procedure a finite number of times.

Notice

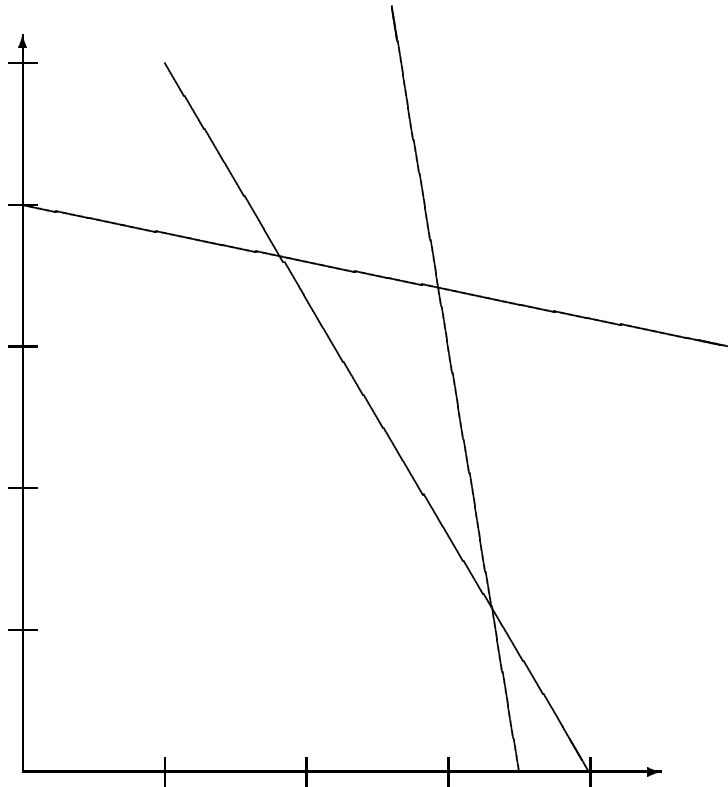
- No stronger inequalities than Chvatal-Gomory exists.
- Even the facet constraints can be generated as Chvatal-Gomory cuts.
- No constructive (polynomial) algorithm for how the linear combination of constraints should be chosen.
- In practice, the derivation of Chvatal-Gomory cuts must rely on specific features of a given application.

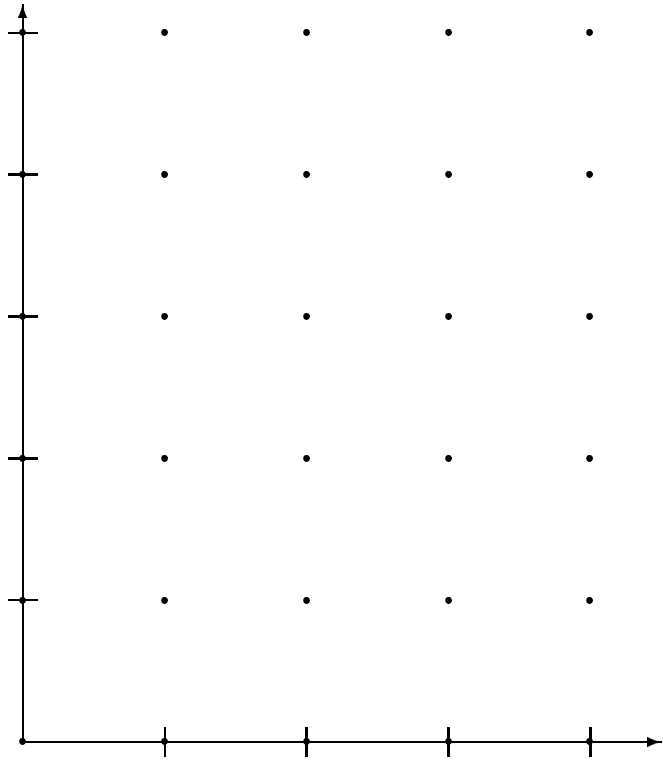
Gomory cuts is a systematical way of deriving cutting planes.

Only 0-1 case

All bounded integer variables can be expressed as sum of binary variables.

The set $X = P \cap \mathbb{Z}^n$





Proof (0-1 case)

$$P = \{x \in \mathbb{R}^n : Ax \leq b, 0 \leq x \leq 1\} \neq \emptyset$$

$$X = P \cap \mathbb{Z}^n$$

Assume that

$$\pi x \leq \pi_0 \quad \text{where } \pi, \pi_0 \text{ integers}$$

is a valid inequality for X . We will show that this inequality can be obtained by using the C-G procedure a finite number of times.

- Step 1: Find a large number $t \in \mathbb{Z}_+$ such that

$$\pi x \leq \pi_0 + t$$

is a valid C-G inequality

- Step 2: Prove that if

$$\pi x \leq \pi_0 + \tau + 1$$

for $\tau \in \mathbb{Z}_+$ is a C-G inequality for X then also

$$\pi x \leq \pi_0 + \tau$$

is a C-G inequality for X .

- Step 3: Use step 2 for $\tau = t, \dots, 0$ each time getting a new C-G inequality

Step 1

The inequality

$$\pi x \leq \pi_0 + t$$

is valid for P for some $t \in \mathbb{Z}_+$.

Proof

- Since $x \leq 1$ we have

$$\pi x \leq \pi 1$$

which can be obtained as C-G inequality using multipliers $u = \pi$.

- Take

$$t = \lceil \pi 1 \rceil - \pi_0$$

Step 2

Difficult part (not proved completely)

a) Prove that if $\pi x \leq \pi_0 + \tau + 1$ with $\tau \in \mathbb{Z}_+$ then

$$\pi x \leq \pi_0 + \tau + \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)$$

is a C-G inequality for X for every partition (N^0, N^1) of $N = \{1, \dots, n\}$.

b) Use partitionings $(T^0 \cup \{n\}, T^1)$ and $(T^0, T^1 \cup \{n\})$ to obtain a new inequality for (T^0, T^1) .

c) Derive all valid inequalities for partitionings of $N' = \{1, \dots, n-1\}$

d) Repeating this procedure n times implies that we eliminate the sums on the right and thus

$$\pi x \leq \pi_0 + \tau$$

Time complexity

- part (c) takes $O(2^n)$,
part (d) is performed n times,
in total $O(n2^n)$
- we run Step 2 $O(t)$ times, thus in total $O(tn2^n)$.

Step 2, a)

Will show that

$$0 \leq \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)$$

is a C-G inequality

- Since $0 \leq x_j$ then

$$0 \leq \sum_{j \in N^0} x_j$$

is a C-G inequality

- Since $x_j \leq 1$ then $0 \leq 1 - x_j$ and thus

$$0 \leq \sum_{j \in N^1} (1 - x_j)$$

- The two inequalities added is a C-G inequality

Step 2, a)

Consider all vertices x of P

- x integer: then $\pi x \leq \pi_0$ (definition of valid inequality)
- x fractional: exists $\epsilon > 0$ such that

$$\epsilon \leq \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)$$

Step 2, a)

Choose $M \geq t/\epsilon$

$$t \leq M\epsilon \leq M \left(\sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j) \right)$$

We already had the inequality

$$\pi x \leq \pi_0 + \tau + 1$$

use weights $1/M$ and $(M - 1)/M$ for the two inequalities getting C-G inequality

$$\pi x \leq \pi_0 + \tau + \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)$$

Step 2, b)

Use partitions $(T^0 \cup \{n\}, T^1)$ and $(T^0, T^1 \cup \{n\})$

$$\pi x \leq \pi_0 + \tau + \sum_{j \in T^0 \cup \{n\}} x_j + \sum_{j \in T^1} (1 - x_j)$$

and

$$\pi x \leq \pi_0 + \tau + \sum_{j \in T^0} x_j + \sum_{j \in T^1 \cup \{n\}} (1 - x_j)$$

using multipliers $1/2$ and $1/2$ we get

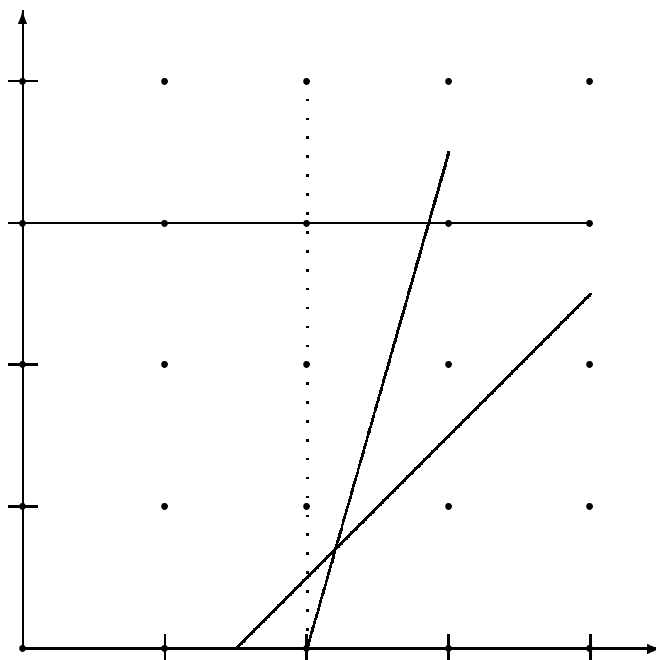
$$\pi x \leq \pi_0 + \tau + \sum_{j \in T^0} x_j + \sum_{j \in T^1} (1 - x_j)$$

Gomory Cuts

- Systematical way of generating valid inequalities
- In each step the current LP-solution will be separated
- Ensures that an integer solution will be reached after a number of steps

Example

$$\begin{array}{ll} \text{maximize} & 4x_1 - x_2 \\ \text{subject to} & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 3 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \geq 0, \text{ integer} \end{array}$$



Gomory Cuts - example

Adding slack variables x_3, x_4, x_5 , and solving LP-problem (Hillier and Lieberman simplex table)

basis	z	x_1	x_2	x_3	x_4	x_5	rhs
z	1			$\frac{4}{7}$	$\frac{1}{7}$		$\frac{59}{7}$
x_1		1		$\frac{1}{7}$	$\frac{2}{7}$		$\frac{20}{7}$
x_2			1		1		3
x_5				$-\frac{2}{7}$	$\frac{10}{7}$	1	$\frac{23}{7}$

The simplex table as equations

$$A_B x_B + A_N x_N = b$$

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$\begin{array}{rcl}
 \max & \frac{59}{7} & + \\
 \text{s.t.} & & \\
 & & - \frac{4}{7}x_3 - \frac{1}{7}x_4 \\
 & x_1 & + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7} \\
 & x_2 & + x_4 = 3 \\
 & & - \frac{2}{7}x_3 + \frac{10}{7}x_4 + x_5 = \frac{23}{7} \\
 & x_1, x_2, x_3, x_4, x_5 & \geq 0, \text{ integer}
 \end{array}$$

The optimal LP-solution is

$$(x_1, x_2, x_3, x_4, x_5) = \left(\frac{20}{7}, 3, 0, 0, \frac{23}{7}\right)$$

which is fractional.

From first equation in Simplex table we get

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = 2 + \frac{6}{7}$$

hence for some $k \in \mathbb{Z}$ we have

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{6}{7} + k$$

from which we get the bound on k

$$k = \frac{1}{7}x_3 + \frac{2}{7}x_4 - \frac{6}{7} \geq -\frac{6}{7}$$

meaning that $k \geq 0$ since $k \in \mathbb{Z}$. Hence we have

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

or substituting the slack variables x_3 and x_4 we get

$$\frac{1}{7}(14 - 7x_1 + 2x_2) + \frac{2}{7}(3 - x_2) \geq \frac{6}{7}$$

which can be reduced to $x_1 \leq 2$.

Modular Arithmetics

Valid inequalities for

$$S = \left\{ x \in \mathbb{Z}_+^n : \sum_{j \in N} a_j x_j = a_0 \right\}$$

Extend S to all points which satisfy the inequality plus kd , where $k > 0$, integer, $d \geq 1$, integer.

$$S_d = \left\{ x \in \mathbb{Z}_+^n : \sum_{j \in N} a_j x_j = a_0 + kd, \text{ some integer } k \right\}$$

Let b_j be the remainder when a_j is divided by d . Thus

$$a_j = b_j + \alpha_j d$$

where $0 \leq b_j < d$ and α_j integer. Then

$$S_d = \left\{ x \in \mathbb{Z}_+^n : \sum_{j \in N} b_j x_j = b_0 + kd, \text{ some integer } k \right\}$$

The integer k must satisfy

$$\begin{aligned} k &= \frac{\sum b_j x_j}{d} - \frac{b_0}{d} \\ &\geq 0 - \frac{b_0}{d} && \text{since } \sum_{j \in N} b_j x_j \geq 0 \\ &> -1 && \text{since } b_0/d < 1 \\ &\geq 0 && \text{since } k \text{ integer} \end{aligned}$$

Thus we have the valid inequality

$$\sum_{j \in N} b_j x_j \geq b_0$$

Since $S \subset S_d$, inequality is valid for S .

Gomory Cuts

Gomory (1963) presented a general technique for solving IP problems

- 1 Solve the LP-relaxation
- 2 Choose one of the basis integer variables taking a fractional value

$$x_i + \sum_{j \in N} a_j x_j = a_0 \quad (1)$$

- 3 Use the corresponding equation to separate the inequality

$$\sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j \geq (a_0 - \lfloor a_0 \rfloor) \quad (2)$$

- 4 Incorporate the new constraint and repeat.

Proposition 1 Inequality (2) is a valid inequality which separates the current LP solution from the feasible set.

Proof [The inequality is valid]. Using modular arithmetics with $d = 1$

$$(1 - \lfloor 1 \rfloor)x_i + \sum_{j \in N} (a_j - \lfloor a_j \rfloor)x_j \geq (a_0 - \lfloor a_0 \rfloor)$$

[Separates current solution]. Current solution was $x_i = a_0$ and $x_j = 0, j \in N$. Inserted in (2)

$$\sum_{j \in N} (a_j - \lfloor a_j \rfloor)0 \geq (a_0 - \lfloor a_0 \rfloor) > 0$$

Gomory Cuts

For pure IP-models we have

Proposition

If we always derive the Gomory cut from the first equation in which the basis variable is fractional, then the algorithm will find an integer optimal solution in a finite number of steps

However

There is no bound on the number of steps

Disjunctive Arithmetics

Proposition Assume that

$$\sum_{j \in N} \pi_j x_j \leq \pi_0$$

is a valid inequality for S_1 and

$$\sum_{j \in N} \pi'_j x_j \leq \pi'_0$$

is a valid inequality for S_2 . Then

$$\sum_{j \in N} \min(\pi_j, \pi'_j) x_j \leq \max(\pi_0, \pi'_0)$$

is a valid inequality for $S_1 \cup S_2$.

Proof If we have the valid inequality

$$\sum_{j \in N} \pi_j x_j \leq \pi_0$$

then also

$$\sum_{j \in N} b_j x_j \leq b_0$$

is a valid inequality if $b_j \leq \pi_j$ and $b_0 \geq \pi_0$. \square

Branch-and-cut algorithms

Combines best properties from Branch-and-bound and cutting plane.

- Basically a branch-and-bound algorithm
- at each node solve LP-relaxation to find bound
- generate valid inequalities which separate the LP-solution, and which are valid for the whole problem
- maintain pool of valid inequalities
- branch when cuts have slow convergence to integrality
- convergence ensured by branch-and-bound
- heuristic generation of cuts
- problem specific cuts

Applications

- General MIP
- Traveling Salesman Problem
- Steiner Tree
- Scheduling
- Graph partitioning