Friday, November 15

Program of the day:

- Solving MIP models by branch-and-bound (Wolsey chapter 7)
- Applications: Knapsack Problem (demo)
- Design issues in branch-and-bound
- Relaxations

Techniques for MIP

- Preprocessing
- Branch-and-bound
- Valid cuts

Development

1960 Brakthrough: branch-and-bound

1970 Small problems ($n < 100$) may be solved. Exponential growth, many important problems cannot be solved.

1983 Crowder, Johnson, Padberg: new algorithm for pure BIP. Sparse matrices up to ($n = 2756$).

1985 Johnson, Kostreva, Sahl: further improvements.

1987 Van Roy, Wolsey: Mixed IP. Up to 1000 binary variables, several continuous variables.

Now Preprocessing, addition of cuts, good branching strategies

Solving IP by enumeration

- Binary IP

$$\begin{array}{lll} \text{maximize} & 2x_1 + 3x_2 - 1x_3 + 5x_4 & \\ \text{subject to} & 4x_1 + 1x_2 + 2x_3 + 3x_4 & \leq 8 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} & \end{array}$$

- Integer IP

$$\begin{array}{lll} \text{maximize} & 2x_1 + 3x_2 - 1x_3 + 5x_4 & \\ \text{subject to} & 4x_1 + 1x_2 + 2x_3 + 3x_4 & \leq 8 \\ & x_1, x_2, x_3, x_4 \in \mathbb{N} & \end{array}$$

- Mixed integer IP

$$\begin{array}{lll} \text{maximize} & 2x_1 + 3x_2 - 1x_3 + 5x_4 & \\ \text{subject to} & 4x_1 + 1x_2 + 2x_3 + 3x_4 & \leq 8 \\ & x_1, x_2 \in \{0, 1\} & \\ & x_3, x_4 \in \mathbb{R} & \end{array}$$

Elements of Branch-and-bound

Problem

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & x \in S \end{array}$$

- **Divide and conquer** (Wolsey prop. 7.1)
  $S = S_1 \cup S_2 \cup \ldots \cup S_K$ and $z^k = \max\{cx : x \in S_k\}$
  $$z = \max_{k=1,\ldots,K} z^k$$
  Often: decomposition by splitting on decision variable
  Overlap between S_i and S_j is allowed

- **Upper bound function** (Wolsey prop. 7.2)
  $$\overline{z}^k = \sup\{cx : x \in S_k\}$$
  then
  $$\overline{z} = \max \overline{z}^k$$
  is an upper bound on S

- **Lower bound** (so far best solution) $\underline{z}$

- **Upper bound test**
  $$\text{if } \overline{z}^k \leq \underline{z} \text{ then } x^* \notin S_k$$

Example: Knapsack Problem

Given n items and a knapsack

- Item j has the weight w_j
- Profit of item j is p_j
- The capacity of the knapsack is c

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} p_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} w_j x_j \leq c \\ & x_j \in \{0,1\}, \quad j = 1, \ldots, n. \end{aligned}$$

Important problem

- Budgeting
- Transportation
- Subproblem (e.g. separation of valid inequalities)

Branch-and-bound

A systematical enumeration technique for solving IP/MIP problems, which apply bounding rules to avoid to examine specific parts of the solution space.

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & x' \geq 0 \\ & x'' \geq 0, \text{integer} \end{array}$$

- Branching tree enumerates all integer variables.
- Once all integer variables are fixed, remaining problem is solved by LP.
- General MIP algorithm does not know structure of problem
- Upper bounds $\overline{z}$ are derived in each node by LP-relaxation.
- If $\overline{z} \leq \underline{z}$ then descendant nodes need not to be examined

Branch-and-bound for MIP

Recursive procedure which at each node:

- If infeasible, backtrack
- Solve LP-relaxation, getting $\overline{x}$ and $\overline{z}$
- If $\overline{z} \leq \underline{z}$ then backtrack
- If all x are integral: update $\underline{z}$, backtrack
- Choose a fractional variable $\overline{x}_i = d$
- Branch on

$$\overline{x}_i \leq \lfloor d \rfloor \qquad\qquad \overline{x}_i \geq \lceil d \rceil$$

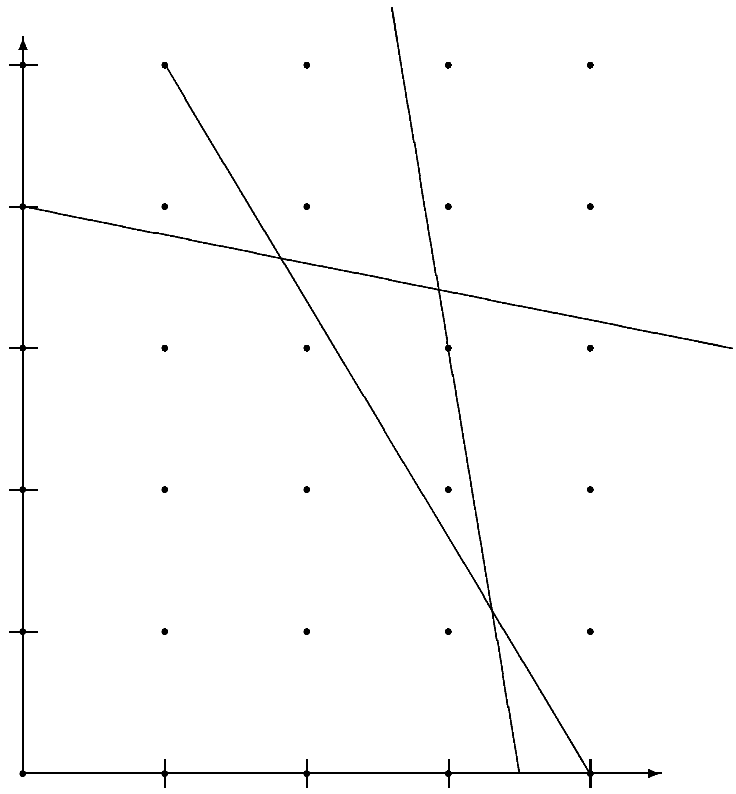
Where

- objective function should be maximized
- $\underline{z}$ is so far best solution (incumbent solution)
- $\overline{z}$ is upper bound at node
- $\overline{x}$ is LP-solution to current problem

Branch-and-bound for MIP

Example:

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & 5x_2 & \leq & 20 \\ & 5x_1 & + & 3x_2 & \leq & 20 \\ & 6x_1 & + & x_2 & \leq & 21 \\ & x_1, & & x_2 & \geq & 0, \text{ integer} \end{array}$$



Branch on most fractional variable, best-first search

Root node

- LP-solution $x_1 = \frac{20}{11} = 1.8181, x_2 = \frac{40}{11} = 3.6363$.
- Lower bound $\underline{z} = -\infty$.
- Two nodes: $x_2 \leq 3$ and $x_2 \geq 4$ with upper bounds $\overline{z} = 5.2$ and $\overline{z} = 4$.

Node 1

- Add constraint $x_2 \leq 3$, getting LP-solution $x_1 = \frac{11}{5} = 2.2$ and $x_2 = 3$.
- Two nodes: $x_1 \leq 2$ and $x_1 \geq 3$ with upper bounds $\overline{z} = 5$ and $\overline{z} = \frac{14}{3} = 4.6667$.

Node 2

- Add constraint $x_1 \leq 2$, getting LP-solution $x_1 = 2$ and $x_2 = 3$. Upper bound $\overline{z} = 5$. Feasible solution $\underline{z} = 5$.

Node 3

- Add constraint $x_1 \geq 3$, getting LP-solution $x_1 = 3$ and $x_2 = \frac{5}{3} = 1.6667$. Upper bound $\overline{z} = 4.6667 < \underline{z}$.

Node 4

- Add constraint $x_2 \geq 4$, getting LP-solution $x_1 = 0$ and $x_2 = 4$. Upper bound $\overline{z} = 4 < \underline{z}$.

Design issues

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & x \in S \end{array}$$

Pruning rules (Wolsey 7.2)

- Prune by optimality $z^k = \max\{cx : x \in S_k\}$
- Prune by bound $\overline{z}_k \leq \underline{z}$
- Prune by infeasibility $S_k = \emptyset$

Branching rules (Wolsey 7.4)

- most fractional variable j i.e. $x_j - \lfloor x_j \rfloor$ close to $\frac{1}{2}$
- least fractional variable
- greedy approach

Selecting next problem

- Depth-first-search
  (quickly find solution, small changes in LP, space)
- Best-first-search
  (greedy approach)

Design issues

Relaxation (Wolsey 2.1)

$$\max\{cx : x \in S\} \qquad (IP)$$
$$\max\{f(x) : x \in T\} \qquad (RP)$$

RP is a relaxation of IP if

- $S \subseteq T$
- $f(x) \geq cx$ for all $x \in S$

Which constraints should be relaxed

- Quality of bound (tightness of relaxation)
- Remaining problem can be solved efficiently
- Constraints difficult to formulate mathematically
- Constraints which are too expensive to write up