Friday, November 8

Program of the day

- Overview of course, exercises
- Introduction to Integer Programming
- Modelling (Williams, chapter 9)
- Applications: Opencast mining
- Evaluation

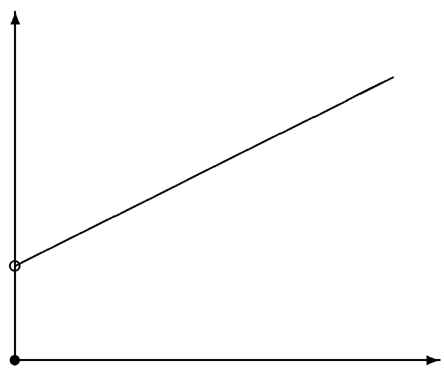
Purpose of the course

- To learn to build complex models from real life using Mathematical Programming
- To know techniques for solving Mathematical Programming models
- To understand that some problems can be solved efficiently and some cannot
- To learn that the same problem may be formulated in different ways, which are easier/harder to solve
- To know a number of techniques for decreasing solution times (or turn a problem from practically "unsolvable" to "solvable")

Integer Programming

In first part of course: continuous variables, linear constraints

- Most products are integral (apart from liquids)
  Airplane production, Tomato Soups.
- Structure of problem leads to IP
  Graph problems.
- Nonlinear objective functions or constraints occur frequently



- Logical conditions
  "If I use vegetable oil in the blend, then I must also add 5ml of preservatives"

Integer Programming

General formulation:

$$\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{1j} x_j \le b_1 \\
& \vdots \\
& \sum_{j=1}^{n} a_{mj} x_j \le b_m \\
& x_j \ge 0, \quad j = 1, \ldots, n, \qquad\qquad x \text{ integer}
\end{aligned}$$

where

- A is a $m \times n$ matrix
- b is a m-vector
- c is a n-vector

- IP: integer programming model
- PIP: pure integer programming model
- MIP: mixed integer programming model

ILP is not ideal model, but bounds from LP (Edmonds)

Integer Programming

IP powerful method for modelling

- LP easy to solve by e.g. Simplex
  (polynomial time by interior-point methods).
- General IP is NP-hard
- Many concrete problems may be solved despite NP-hardness
- Specific techniques for individual problems

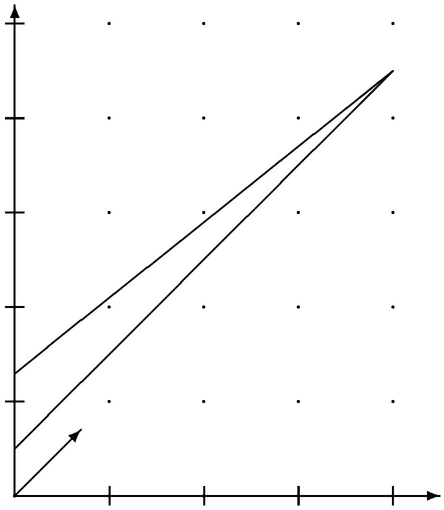
Special problems

- travelling salesman problem
- project selection
- transportation problem
- assignment problem
- assembly line balancing
- set partitioning problem
- aircrew scheduling
- depot location problem
- sequencing problem
- job-shop scheduling

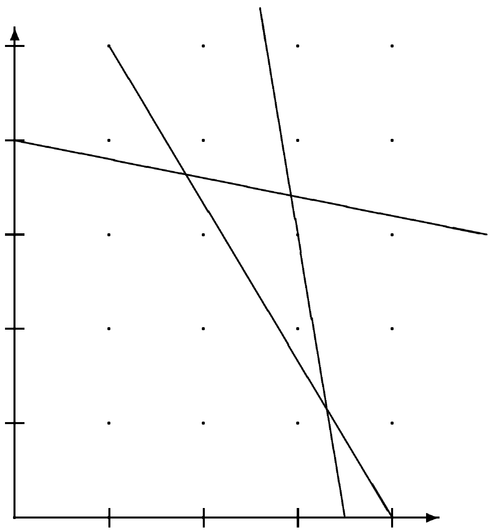
Hardness of IP

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & -2x_1 + \quad 2x_2 \geq 1 \\ & -8x_1 + 10x_2 \leq 13 \\ & x_1, x_2 \geq 0, \text{integer} \end{array}$$

Solutions are not found in extreme points (or nearby)



Find convex hull



Model building

- Indicator variables
- Non-convex problems
- Nonlinear fuctions
- Logical expressions
- Transformation of “human text” to ILP

Indicator variables

- Most important modelling tool!
- $\delta \in \{0, 1\}$
- $\delta = 1$ if and only if some event happens.

Model:

$$\delta = 1 \;\Leftrightarrow\; x > 0$$
$$\delta \in \{0, 1\}, \quad x \geq 0$$

$\boxed{\delta = 1 \;\Rightarrow\; x > 0}$

$\delta = 1 \Rightarrow x \geq \epsilon$ $\qquad$ ϵ level for x regarded as 0

$x - \epsilon\delta \geq 0,\; \delta \in \{0, 1\}$

$\boxed{x > 0 \;\Rightarrow\; \delta = 1}$

$\delta = 0 \;\Rightarrow\; x = 0$

$x - M\delta \leq 0,\; \delta \in \{0, 1\}$ $\qquad$ M upper bound on x

Indicator variables

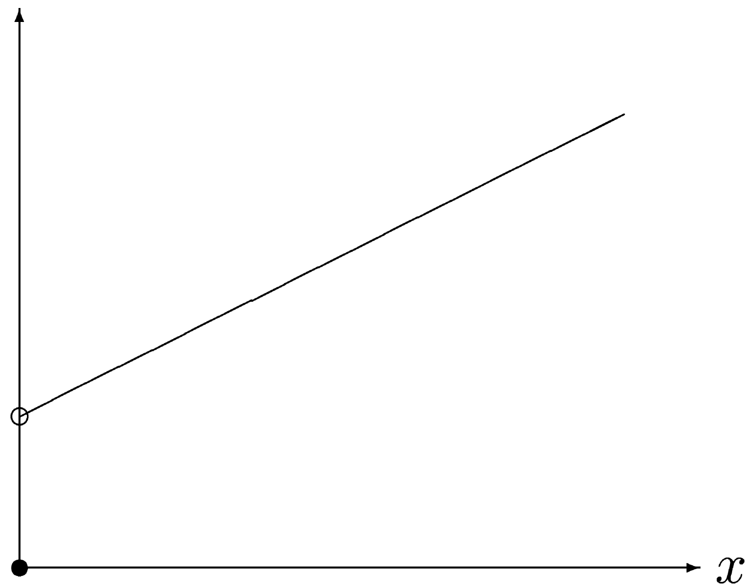
Logical implications $\boxed{X \;\Leftrightarrow\; Y}$

X	Y	$X \Rightarrow Y$	$X \Leftarrow Y$	$\neg X \Rightarrow \neg Y$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Fixed-charge problem

cost function

$$f(x) = \begin{cases} ax + b & \text{if } \; x > 0 \\ 0 & \text{if } \; x = 0 \end{cases}$$



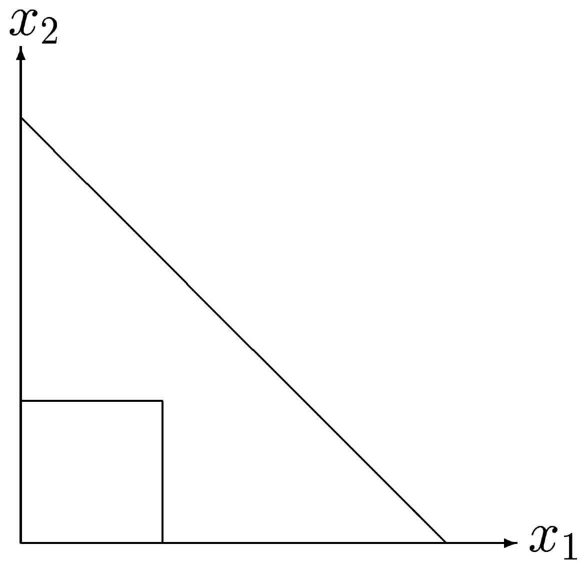
Model:

$$\begin{cases} \text{minimize} & ax + \delta b \\ \text{subject to} & x - M\delta \leq 0 \\ & x - \epsilon\delta \geq 0 \\ & \delta \in \{0, 1\}, \quad x \geq 0 \end{cases}$$

Non-convex problems

constraints:

$$x_1 + x_2 \leq b$$
$$x_1 \geq 1 \text{ or } x_2 \geq 1$$
$$x_1, x_2 \geq 0$$



Modeling tool

$$\delta = 1 \Rightarrow x \geq \epsilon$$

Two indicator variables δ_1, δ_2:

$$\begin{cases} x_1 + x_2 & \leq & b \\ \delta_1 + \delta_2 & \geq & 1 \\ x_1 - 1\delta_1 & \geq & 0 \\ x_2 - 1\delta_2 & \geq & 0 \\ x_1, x_2 & \geq & 0, \\ \delta_1, \delta_2 & \in & \{0, 1\} \end{cases}$$

Indicator variables

"A is included in the blend iff B is included in the blend"

can be modeled by using constraints

$x_A > 0 \;\Rightarrow\; \delta = 1$	$x_A - M\delta \leq 0, \quad \delta \in \{0,1\}$ M upper bound on x_A
$\delta = 1 \;\Rightarrow\; x_B > 0$	$x_B - \epsilon\delta \geq 0, \quad \delta \in \{0,1\}$ ϵ level for x_B regarded as 0

Example

Assume that x_A and x_B are proportions in blend i.e. $x_A + x_B = 1$.

$$M = 1 \qquad \epsilon = 0.01$$

Formulation:

$$\begin{cases} x_A - \delta & \leq \; 0 \\ x_B - 0.01\delta & \geq \; 0 \\ \delta & \in \; \{0,1\} \end{cases}$$

Indicator variables for linear inequalities

Example

"If resources needed for production of x_1, x_2 and x_3 are below the limit of one truck, then use the other truck for some other purpose."

General form

$$\boxed{\sum_{j=1}^{n} a_j x_j \leq b \quad \Leftrightarrow \quad \delta = 1, \quad \delta \in \{0, 1\}}$$

- $\sum_{j=1}^{n} a_j x_j \leq b \Leftarrow \delta = 1$ has the MIP formulation

$$\boxed{\sum_{j=1}^{n} a_j x_j + M\delta \leq M + b}$$

where M is upper bound on $\sum_{j=1}^{n} a_j x_j - b$

$\delta = 1$: $\sum_{j=1}^{n} a_j x_j \leq b$
$\delta = 0$: $\sum_{j=1}^{n} a_j x_j - b \leq M$

Indicator variables for linear inequalities

- $\sum_{j=1}^{n} a_j x_j \leq b \Rightarrow \delta = 1$ has the MIP formulation

$$\boxed{\sum_{j=1}^{n} a_j x_j - (m - \epsilon)\delta \geq b + \epsilon}$$

  where m is lower bound on $\sum_{j=1}^{n} a_j x_j - b$.

$$\delta = 0 \Rightarrow \sum_{j=1}^{n} a_j x_j \geq b + \epsilon$$

$\delta = 0$: $\sum_{j=1}^{n} a_j x_j \geq b + \epsilon$

$\delta = 1$: $\sum_{j=1}^{n} a_j x_j - m + \epsilon \geq b + \epsilon$

$\sum_{j=1}^{n} a_j x_j - b \geq m$

Indicator variables for inequalities, example

Logical condition

$$\begin{array}{l} 2x_1 + 3x_2 \leq 1 \Leftrightarrow \delta = 1 \\ \delta \in \{0, 1\} \\ 0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1 \end{array}$$

We find

$$\begin{array}{rcl} M & = & \text{u.b.}(\Sigma_{j=1}^{n} a_j x_j - b) \\ & = & \text{u.b.}(2x_1 + 3x_2 - 1) = 4 \end{array}$$

and

$$\begin{array}{rcl} m & = & \text{l.b.}(\Sigma_{j=1}^{n} a_j x_j - b) \\ & = & \text{l.b.}(2x_1 + 3x_2 - 1) = -1 \end{array}$$

choose $\epsilon = 0.01$, i.e. constraint broken when $2x_1 + 3x_2 \geq 1.01$

Constraints

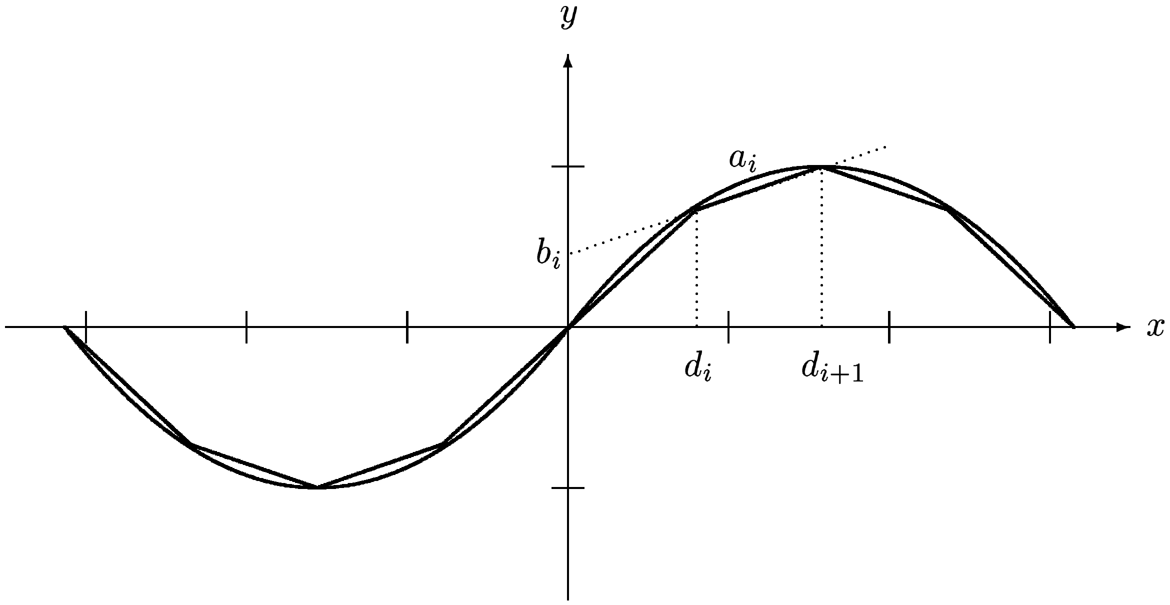
$$\begin{array}{l} 2x_1 + 3x_2 + 4\delta \leq 4 + 1 \\ 2x_1 + 3x_2 - (-1 - 0.01)\delta \geq 1 + 0.01 \end{array}$$

Which results in model:

$$\begin{cases} 2x_1 + 3x_2 + 4\delta \leq 5 \\ 2x_1 + 3x_2 + 1.01\delta \geq 1.01 \\ \delta \in \{0, 1\} \\ 0 \leq x_1 \leq 1, \\ 0 \leq x_2 \leq 1 \end{cases}$$

Nonlinear functions

Frequently, the objective function or some of the constraints may contain nonlinear functions.



Approx. nonlinear function by piecewise linear function

- Split into m intervals
- For each interval $[d_i, d_{i+1}]$

$$d_i \leq x \leq d_{i+1} \;\Leftrightarrow\; y = a_i x + b_i$$

- Model as

$$\begin{cases} d_i \leq x & \Leftrightarrow \quad \delta_1 = 1 \\ x \leq d_{i+1} & \Leftrightarrow \quad \delta_2 = 1 \\ \delta_1 + \delta_2 = 2 & \Leftrightarrow \quad \delta = 1 \\ y = a_i x + b_i & \Leftrightarrow \quad \delta = 1 \end{cases}$$

- Many intervals m, better precision
  but much harder to solve!

Logical conditions and 0-1 variables

a) *If no depot is sited here then it will not be possible to supply any of the customers from the depot.*

b) *If we manufacture product A then we must also manufacture product B or at least one of products C and D.*

c) *If we do not place an electronic module in this position, then no wires can be connected into this position.*

Introduce an indicator variable $\delta_i \in \{0, 1\}$ with each condition X_i

$$\boxed{\text{condition } X_i \text{ is true} \Leftrightarrow \delta_i = 1}$$

In this way we may formulate:

$X_1 \vee X_2$	$\delta_1 + \delta_2 \geq 1$
$X_1 \wedge X_2$	$\delta_1 = 1, \delta_2 = 1$
$X_1 \Rightarrow X_2$	$\delta_1 - \delta_2 \leq 0$
$X_1 \Leftrightarrow X_2$	$\delta_1 - \delta_2 = 0$

Transformation to linear form

Write up the text in ordinary mathematical form

$$(\sin(x_1) \le \tfrac{1}{2} \ \lor \ x_1 x_2 \le x_3) \Rightarrow (x_3 = 1 \lor x_2 + x_1 \le 1)$$

Stepwise transformation

1 Arithmetic functions are replaced by piecewise linear approximations of the functions.

2 Products of decision variables are transformed into products of binary variables. Products of binary variables may easily be expressed as logical constraints, and thus put on binary form.

3 Relations are transformed into linear inequalities with boolean variables.

$$(ax \le b) \Leftrightarrow (\delta = 1)$$

4 Boolean logics are transformed into linear form.

$$(B_1 \lor B_2) \Leftrightarrow (\delta' = 1)$$

5 The resulting expression should be true $\delta_{all} = 1$

6 Domains of variables are defined.

$$\delta_i \in \{0, 1\}$$

Transformation of general constraints to linear form

Step 3:

Relation	ILP-constraints	
$Ax \leq b$	$Ax + (\delta - 1)M \leq b,$	$Ax + \delta M \geq b + \varepsilon$
$Ax < b$	$Ax + (\delta - 1)M \leq b - \varepsilon,$	$Ax + \delta M \geq b$
$Ax > b$	$Ax + (1 - \delta)M \geq b + \varepsilon,$	$Ax - \delta M \leq b$
$Ax \geq b$	$Ax + (1 - \delta)M \geq b,$	$Ax - \delta M \leq b - \varepsilon$
$Ax = b$	$Ax \geq b \ \wedge \ Ax \leq b,$	

Step 4:

Relation	Meaning	ILP-constraints
$B_1 \vee B_2$	$\delta = 1 \ \Leftrightarrow \ \delta_1 = 1 \vee \delta_2 = 1$	$\delta - \delta_1 - \delta_2 \leq 0, \quad \delta_1 + \delta_2 - 2\delta \leq 0$
$B_1 \wedge B_2$	$\delta = 1 \ \Leftrightarrow \ \delta_1 = 1 \wedge \delta_2 = 1$	$2\delta - \delta_1 - \delta_2 \leq 0, \quad \delta_1 + \delta_2 - \delta \leq 1$
$B_1 \ \Rightarrow \ B_2$	$\delta = 1 \ \Leftrightarrow \ (\delta_1 = 1 \Rightarrow \delta_2 = 1)$	$\delta_1 - \delta_2 + \delta \leq 1, \quad \delta_1 - \delta_2 + 2\delta \geq 1$
$B_1 \ \Leftrightarrow \ B_2$	$\delta = 1 \ \Leftrightarrow \ (\delta_1 = 1 \ \Leftrightarrow \ \delta_2 = 1)$	use: $(B_1 \ \Rightarrow \ B_2) \ \wedge \ (B_2 \ \Rightarrow \ B_1)$
$\neg B_1$	$\delta = 1 \ \Leftrightarrow \ \neg(\delta_1 = 1)$	$\delta = 1 - \delta_1$

Opencast mining (williams example 12.15)

Concentration c_{ijk} of pure metal for each block

- Level 1 (surface)

1.5	1.5	1.5	0.75
1.5	2.0	1.5	0.75
1.0	1.0	0.75	0.50
1.5	1.5	1.5	0.25

- Level 2 (25 ft depth)

4.0	4.0	2.0
3.0	3.0	1.0
2.0	2.0	0.5

- Level 3 (50 ft depth)

12.0	6.0
5.0	4.0

- Level 4 (75 ft depth)

6.0

Cost of extraction

Level 1: $e_1 = 3.000$ pounds
Level 2: $e_2 = 6.000$ pounds
Level 3: $e_3 = 8.000$ pounds
Level 4: $e_4 = 10.000$ pounds

Revenue from 100% block is $R = 200.000$ pounds

Opencast mining

Introduce variables x_{ijk}

- Level 1 (surface)

x_{111}	x_{112}	x_{113}	x_{114}
x_{121}	x_{122}	x_{123}	x_{124}
x_{131}	x_{132}	x_{133}	x_{134}
x_{141}	x_{142}	x_{143}	x_{144}

- Level 2 (25 ft depth)

x_{211}	x_{212}	x_{213}
x_{221}	x_{222}	x_{223}
x_{231}	x_{232}	x_{233}

- Level 3 (50 ft depth)

x_{311}	x_{312}
x_{321}	x_{322}

- Level 4 (75 ft depth)

x_{411}

Maximize net profit

$$\sum(R \cdot c_{ijk} - e_i)x_{ijk}$$

Constraints

$$\delta_{ijk} = 1 \Rightarrow \begin{cases} \delta_{i-1,j,k} & = 1 \\ \delta_{i-1,j-1,k} & = 1 \\ \delta_{i-1,j,k-1} & = 1 \\ \delta_{i-1,j-1,k-1} & = 1 \end{cases}$$

All variables $x_{ijk} \in \{0, 1\}$.

Opencast mining

The constraints

$$\delta_{ijk} = 1 \Rightarrow \begin{cases} \delta_{i-1,j,k} & = 1 \\ \delta_{i-1,j-1,k} & = 1 \\ \delta_{i-1,j,k-1} & = 1 \\ \delta_{i-1,j-1,k-1} & = 1 \end{cases}$$

can be expressed as

$$\begin{array}{ll} \delta_{ijk} - \delta_{i-1,j,k} & \leq 0 \\ \delta_{ijk} - \delta_{i-1,j-1,k} & \leq 0 \\ \delta_{ijk} - \delta_{i-1,j,k-1} & \leq 0 \\ \delta_{ijk} - \delta_{i-1,j-1,k-1} & \leq 0 \end{array}$$

Model with 30 binary variables, 56 constraints
Brute-force solution: 2^{30} steps

Opencast mining

$$
\begin{pmatrix}
-1 & & & & & 1 & & & & & & \\
& -1 & & & & 1 & & & & & & \\
& & -1 & & & 1 & & & & & & \\
& & & -1 & & 1 & & & & & & \\
& -1 & & & & & 1 & & & & & \\
& & -1 & & & & 1 & & & & & \\
& & & -1 & & & 1 & & & & & \\
& & & & -1 & & 1 & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & -1 & & & & 1 \\
& & & & & & & & -1 & & & 1 \\
& & & & & & & & & -1 & & 1 \\
& & & & & & & & & & -1 & 1
\end{pmatrix}
$$

The constraint matrix A is totally unimodular (TU)

Solving

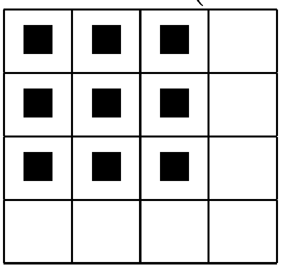
$$
\begin{array}{ll}
\max & c\delta \\
\text{s.t.} & A\delta \leq b \\
 & \delta \geq 0
\end{array}
$$

gives integer solutions for any integer vector b and any c

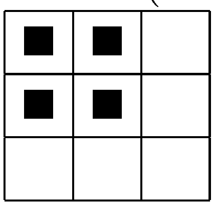
Opencast mining

The optimal solution

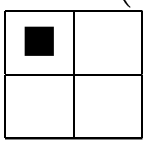
- Level 1 (surface)

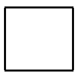
- Level 2 (25 ft depth)

- Level 3 (50 ft depth)

- Level 4 (75 ft depth)

Net profit is 17.500 pounds.