Introduction to Optimization:

Written Exam, 15 December 2000

Your assignment

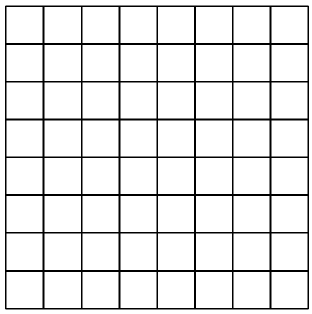
20 different questions Q1-Q20 are posed on the subsequent pages. Q1-Q8 and Q11-Q18 are *multiple choice questions.* For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation to write, for example, "7.b" as your answer to question Q7. Q9-Q10 and Q19-Q20 are ordinary *text questions.* Each correct answer to a multiple choice question gives 4 points, to a text question gives 9 points. The maximum score is thus 100 points.

Note: only the last 10 questions are available

Cutting down the weight — with Gomory cuts

Biotech Ltd. is also producing two pills for slimming. Pill number 1 is a serious product which however has no commercial value. Pill number 2 is a pure plasebo product which due to its pink color is selling very well — and some people even report magnificant results with it. The president of Biotech Ltd. knowns that the "active ingredients" in pill 2 is its very high price which makes people starve to pay the monthly costs. Thus to maximize the production of pill 2 he constructs the following model

$$\begin{array}{lrcrcl} \text{maximize} & & & x_2 & & \\ \text{subject to} & 3x_1 & + & 2x_2 & \leq & 6 \\ & -3x_1 & + & 2x_2 & \leq & 0 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0, \text{integer}} \end{array} \quad (1)$$



Q 11 Classify the four constraints

$$\begin{array}{ll} \text{(i)} & 3x_1 + 2x_2 \leq 6 \\ \text{(ii)} & -3x_1 + 2x_2 \leq 0 \\ \text{(iii)} & x_1 \geq 0 \\ \text{(iv)} & x_2 \geq 0 \end{array}$$

according to their strength

11.a) all constraints (i) to (iv) are facet defining.

11.b) constraint (iii) and (iv) are facet defining.

11.c) constraint (i) is redundant and constraint (iv) is facet defining.

11.d) constraint (iii) is redundant and constraint (iv) is facet defining.

11.e) constraint (iii) and (iv) are redundant.

Q 12 In order to solve the LP-relaxation of the model, slack variables x_3 and x_4 are added to the two constraints. The problem is solved using the simplex algorithm. What is the form of the final simplex tableau

12.a) $$\begin{cases} z = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\ x_1 = 1 - \frac{1}{6}x_3 + \frac{1}{6}x_4 \\ x_2 = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \end{cases}$$

12.b) $$\begin{cases} z = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\ x_1 = \frac{3}{2} - \frac{1}{4}x_3 + \frac{1}{4}x_4 \\ x_2 = 1 - \frac{1}{6}x_3 - \frac{1}{6}x_4 \end{cases}$$

12.c) $$\begin{cases} z = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\ x_1 = 1 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \\ x_2 = \frac{3}{2} + \frac{1}{4}x_3 + \frac{1}{4}x_4 \end{cases}$$

12.d) $$\begin{cases} z = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4 \\ x_1 = \frac{3}{2} + \frac{1}{4}x_3 - \frac{1}{4}x_4 \\ x_2 = 1 + \frac{1}{6}x_3 - \frac{1}{6}x_4 \end{cases}$$

Q 13 The first equation $z = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4$ contains a fractional variable, thus a Gomory cut can be derived from this equation. What is the form of the derived inequality if we use the definition of Gomory cuts from Wolsey?

13.a) $\frac{1}{4}x_3 - \frac{1}{4}x_4 \geq -\frac{1}{2}$

13.b) $x_1 \leq 4$

13.c) $x_3 + x_4 \geq 8$

13.d) $x_1 + x_2 \leq 1$

13.e) $x_3 + x_4 \geq 2$

13.f) $x_1 + x_2 \geq 4$

Q 14 Lagrangian relaxing the second constraint $-3x_1 + 2x_2 \leq 0$ in problem (1) leads to a problem LR_λ with only one constraint. What is the objective function of LR_λ

14.a) $x_2 - 3x_1 + \lambda$, where $\lambda \leq 0$

14.b) $(2\lambda + 1)x_2 - 3\lambda x_1$, where $\lambda \geq 0$

14.c) $3\lambda x_1 - 3x_2$, where $\lambda \geq 0$

14.d) $3\lambda x_1 + (1 - 2\lambda)x_2$, where $\lambda \geq 0$

14.e) $3\lambda x_1 - 3x_2$, where $\lambda \leq 0$

14.f) $-3\lambda x_1 - (1 - 2\lambda)x_2$, where $\lambda \leq 0$

Q 15 For which value of λ do we get the tightest upper bound by solving the Lagrangian relaxed problem LR_λ

15.a) $\lambda = 0$

15.b) $\lambda = 1$

15.c) $\lambda = 2$

15.d) $\lambda = \frac{1}{5}$

15.e) $\lambda = \frac{1}{4}$

15.f) $\lambda = \frac{1}{3}$

Uncovering the "Lederhosen" gene

The research lab in Biotech Ltd. is working on a complete classification of the human genes. In particular they are interested in describing the till now unknown "Lederhosen" gene which can be found in certain mountain tribes in Austria. Not surprising this gene can be described by solving a two-constrained knapsack problem:

$$\begin{array}{llllllllll} \text{maximize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\ \text{subject to} & 5x_1 & - & 6x_2 & + & 5x_3 & + & 8x_4 & \leq & 5 \\ & 7x_1 & + & 3x_2 & + & 4x_3 & + & 3x_4 & \leq & 9 \\ & x_1 & & & & & & & \leq & 1 \\ & & & x_2 & & & & & \leq & 1 \\ & & & & & x_3 & & & \leq & 1 \\ & & & & & & & x_4 & \leq & 1 \\ & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \geq 0, \text{ integer}} \end{array}$$

Q 16 The LP-relaxation of the problem is solved, giving the dual solution $y_1 = \frac{1}{17}$, $y_2 = \frac{3}{17}$, $y_3 = 0$, $y_4 = \frac{14}{17}$, $y_5 = 0$, $y_6 = 0$. What is the primal solution

16.a) $x_1 = 0$, $x_2 = 1$, $x_3 = \frac{15}{17}$, $x_4 = \frac{14}{17}$
16.b) $x_1 = \frac{15}{17}$, $x_2 = 0$, $x_3 = 1$, $x_4 = \frac{14}{17}$
16.c) $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{1}{17}$, $x_4 = \frac{1}{17}$
16.d) $x_1 = \frac{15}{17}$, $x_2 = \frac{14}{17}$, $x_3 = 0$, $x_4 = 1$
16.e) $x_1 = \frac{14}{17}$, $x_2 = \frac{15}{17}$, $x_3 = 1$, $x_4 = 0$

Q 17 Separate the most violated cover inequality for the first constraint

$$5x_1 - 6x_2 + 5x_3 + 8x_4 \leq 5$$

17.a) $x_1 + x_2 + x_3 \leq 2$
17.b) $x_1 + x_2 + x_3 \leq 1$
17.c) $x_1 - x_2 + x_3 + x_4 \leq 3$
17.d) $x_3 + x_4 \leq 1$
17.e) $x_1 + x_2 + x_3 + x_4 \leq 2$
17.f) $x_4 - x_2 \leq 0$.

(Hint: be careful when solving the separation problem since not all the coefficients are positive)

Q 18 Consider now the second inequality

$$7x_1 + 3x_2 + 4x_3 + 3x_4 \leq 9$$

It is obvious that $C = \{2, 3, 4\}$ is a cover, giving the valid inequality $x_2 + x_3 + x_4 \leq 2$. We would however like to lift the cover inequality. What is the maximum value α for which

$$\alpha x_1 + x_2 + x_3 + x_4 \leq 2$$

is a valid inequality.

18.a) $\alpha = 3$

18.b) $\alpha = 2$

18.c) $\alpha = \frac{3}{2}$

18.d) $\alpha = 1$

18.e) $\alpha = \frac{1}{7}$

18.f) $\alpha = 0$

Minimizing the work-load of the raindeers (text question)

Santa Claus is planning the distribution of Christmas gifts. There are three types of gifts corresponding to how the children behaved during the year: A (very well), B (quite well) and C (not so well). For obvious reasons gifts of type A are the largest and most exciting, while type C are consolation prizes.

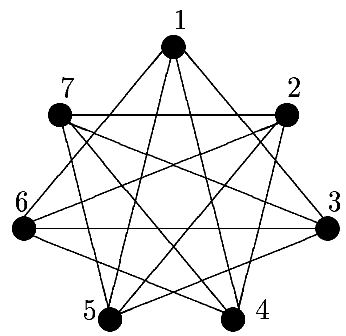
Exactly 1000 gifts should be brought out, but to avoid that too many children become envious, the amount of type A and C may differ by at most 100 gifts. If at least 500 gifts of type B are brought out, then at least 100 gifts of type A must be brought out also.

For bringing out all the gifts, one sleigh is needed pr. 100 gifts of type A, or one sleigh pr. 200 gifts of type B, or one sleigh pr. 300 gifts of type C. To avoid confusion during the hectic hours of Christmas Eve, no sleigh may contain more than one type of gifts. Santa Claus wishes to find a feasible solution using least possible sleighs.

Q 19 Formulate the problem as a Mixed Integer Programming model.

Covering the star (text question)

The vertex cover problem asks to choose a minimal number of nodes such that every edge is covered by at least one node.



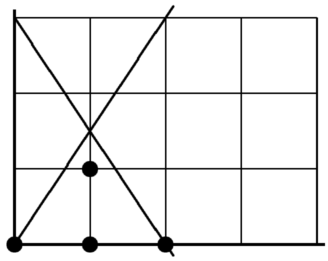
Q 20 Let the variable $x_i \in \{0, 1\}$ indicate whether node i is selected. Formulate the problem as an integer programming model. For the graph shown above, derive the inequality

$$\sum_{i=1}^{7} x_i \geq 5$$

as a Chvatal-Gomory cut.

Answers

Answer 11 Drawing the constraints and the integer points in the set, one gets



It is easy to see that (i) and (ii) are certainly not facet defining, while constraint (iv) is facet defining. Finally (iii) is redundant, since it could be removed without changing the solution set. Thus the correct answer is 11.d). ■

Answer 12 We add the slack variables getting the tableau

$$\begin{aligned} z &= x_2 \\ x_3 &= 6 - 3x_1 - 2x_2 \\ x_4 &= 0 + 3x_1 - 2x_2 \end{aligned}$$

After two pivot operations we get

$$\begin{aligned} z &= \tfrac{3}{2} - \tfrac{1}{4}x_3 - \tfrac{1}{4}x_4 \\ x_1 &= 1 - \tfrac{1}{6}x_3 + \tfrac{1}{6}x_4 \\ x_2 &= \tfrac{3}{2} - \tfrac{1}{4}x_3 - \tfrac{1}{4}x_4 \end{aligned}$$

Thus answer 12.a) is correct. One could also notice that sinze $z = x_2$ the two expressions defining z and x_2 must be equal, and thus only 12.a) can be correct. ■

Answer 13 From the equation $z + \frac{1}{4}x_3 + \frac{1}{4}x_4 = \frac{3}{2}$ we get the Gomory cut as described in Wolsey, Section 8.6. This gives

$$\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

Multiplying by 4 we get $x_3 + x_4 \geq 2$, thus answer 13.e) is correct.

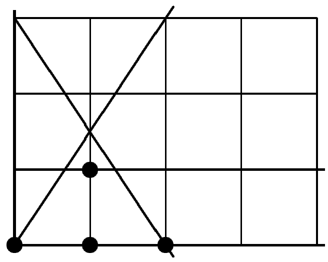
Introducing the original variables in place of $x_3 = 6 - 3x_1 - 2x_2$ and $x_4 = 3x_1 - 2x_2$ we get

$$(6 - 3x_1 - 2x_2) + (3x_1 - 2x_2) \geq 2$$

which may be reduced to

$$x_2 \leq 1$$

Adding the inequality to the problem one gets the solution space



The LP-optimal solution now becomes $x_1 = \frac{4}{3}$ and $x_2 = 1$ with objective value $z = 1$. ■

Answer 14 Lagrangian relaxing the second constraint $-3x_1 + 2x_2 \leq 0$ in (1) using a multiplier $\lambda \geq 0$ gives the objective function

$$x_2 - \lambda(-3x_1 + 2x_2 - 0) = x_2 + 3\lambda x_1 - 2\lambda x_2 = 3\lambda x_1 + (1 - 2\lambda)x_2$$

Thus answer 14.d) is correct. ■

Answer 15 The remaining constraints define the set

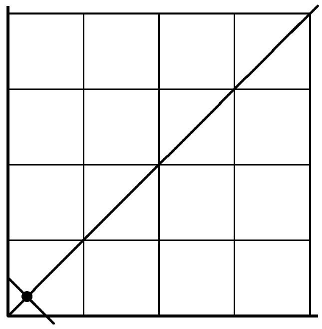
$$X = \left\{ \begin{array}{l} -3x_1 + 2x_2 \leq 0 \\ x_1, x_2 \geq 0, \text{ integer} \end{array} \right\}$$

where we notice that conv$(X) = X$. In this situation Corrolary page 173 in Wolsey say that the best lagrangian relaxation is as strong as the LP-relaxation. Thus we know that the best Lagrangian multiplier λ corresponds to the dual variable of the original problem (1) solved as an LP-problem.

The dual problem is

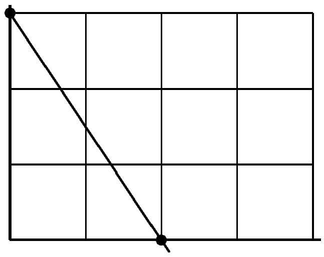
$$\begin{array}{lrcrcl} \text{minimize} & 6y_1 & & & & \\ \text{subject to} & 3y_1 & - & 3y_2 & \geq & 0 \\ & 2y_1 & + & 2y_2 & \geq & 1 \\ & y_1, y_2 \geq 0 & & & & \end{array}$$

where the LP-optimal solution may be found graphically as $y_1 = y_2 = \frac{1}{4}$.



Thus the optimal choice of the Lagrangian multiplier is $\lambda = y_2 = \frac{1}{4}$, making answer 15.e) correct.

One could also try all the proposed values of λ and see which of them results in the tightest bound. This can be done graphically as the set of feasible solutions is independant of the value of λ:



For each of the values of λ we find

$$\begin{array}{llll} \lambda = 0: & z = x_2, & x^* = (0,3), & z^* = 6 \\ \lambda = 1: & z = 3x_1 - x_2, & x^* = (2,0), & z^* = 6 \\ \lambda = 2: & z = 6x_1 - 3x_2, & x^* = (2,0), & z^* = 12 \\ \lambda = \frac{1}{5}: & z = \frac{3}{5}x_1 + \frac{2}{5}x_2, & x^* = (0,3), & z^* = \frac{9}{5} \\ \lambda = \frac{1}{4}: & z = \frac{3}{4}x_1 + \frac{1}{2}x_2, & x^* = (2,0) \vee (0,3), & z^* = \frac{3}{2} \\ \lambda = \frac{1}{3}: & z = x_1 + \frac{1}{3}x_2, & x^* = (2,0), & z^* = 2 \end{array}$$

■

Answer 16 Using complementary slackness we know that the dual constraint is zero if and only if the constraint is not binding. Since $y_4 \neq 0$ constraint 4 must be binding, thus $x_2 = 1$. This leaves only one possible answer 16.a). ■

Answer 17 To separate the most violated cover inequality from

$$5x_1 - 6x_2 + 5x_3 + 8x_4 \leq 5$$

we use the procedure described in Wolsey page 150. Thus first we need to ensure that all coefficients are positive by substituting $x_2' = 1 - x_2$, getting

$$5x_1 + 6x_2' + 5x_3 + 8x_4 \leq 11$$

we solve the separation problem

$$\begin{aligned} \gamma = \text{minimize} \quad & \sum_{i \in I} (1 - x_i)\delta_i \\ \text{subject to} \quad & \sum_{i \in I} a_i \delta_i \geq b + 1 \\ & \delta_i \in \{0, 1\}, \ i \in I. \end{aligned}$$

In the current solution we have $x_2 = 1$ and thus $x_2' = 0$ meaning that the separation problem becomes

$$\begin{aligned} \text{minimize} \quad & 1\delta_1 + 1\delta_2 + \tfrac{2}{17}\delta_3 + \tfrac{3}{17}\delta_4 \\ \text{subject to} \quad & 5\delta_1 + 6\delta_2 + 5\delta_3 + 8\delta_4 \geq 12 \\ & \delta_i \in \{0, 1\}, \ i = 1, \ldots, 4 \end{aligned}$$

with optimal solution $\delta_1 = 0$, $\delta_2 = 0$, $\delta_3 = 1$, $\delta_4 = 1$. As the objective function is $\gamma = \frac{5}{17} < 1$ we have separated a cover inequality with cover $C = \{3, 4\}$. The inequality is

$$x_3 + x_4 \leq |C| - 1 = 1$$

Thus 17.d) is correct.

Adding the new inequality to the problem we get the solution value $z = \frac{17}{7} = 2.429$ The original solution value was $z = \frac{46}{17} = 2.706$, so the cut did have a considerable effect. ■

Answer 18 Using the algorithm described page 149 in Wolsey, we solve the knapsack problem

$$\begin{aligned} \gamma = \text{maximize} \quad & x_2 + x_3 + x_4 \\ \text{subject to} \quad & 3x_2 + 4x_3 + 3x_4 \leq 9 - 7 \\ & x_i \in \{0, 1\}, \ i = 1, \ldots, 4 \end{aligned}$$

with optimal solution $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ and objective value $\gamma = 0$. This means that we may set $\alpha = |C| - 1 - \gamma = 3 - 1 - 0 = 2$. The correct answer is 18.b).

Adding the new inequality to the problem together with the previous cover inequality we get the solution value $z = 2$ and integer solutions $x_2 = x_3 = 1$. ■

Answer 19 We introduce the variables x_a, x_b, x_c to denote the amount of gifts of type A, B and C respectively. Moreover, we use s_a, s_b, s_c to denote the number of sleighs needed for transporting each of the gift types. This immediately gives the formulation

$$\begin{array}{ll} \text{minimize} & s_a + s_b + s_c \\ \text{subject to} & x_a + x_b + x_c = 1000 \\ & 100 s_a - x_a \geq 0 \\ & 200 s_b - x_b \geq 0 \\ & 300 s_c - x_c \geq 0 \\ & x_a - x_c \leq 100 \\ & x_c - x_a \leq 100 \\ & x_b - 501\delta \leq 499 \\ & x_a - 100\delta \geq 0 \\ & x_a, x_b, x_c \geq 0 \\ & s_a, s_b, s_c \geq 0, \text{ integer} \\ & \delta \in \{0,1\} \end{array}$$

The first constraint ensures that 1000 gifts will be brought out. The three following constraints binds the number of sleighs to the number of gifts for each of the three gift types. The next two constraints ensure that the amount of gifts for type A and C do not differ by more than 100. Finally the last inequalities ensure that if $x_b \geq 500$ then $\delta = 1$, and again if $\delta = 1$ then $x_a \geq 100$.

The optimal solution (which was not asked for) is

$$s_a = 2, s_b = 3, s_c = 1, x_a = 150, x_b = 600, x_c = 250, \delta = 1$$

thus six sleighs are needed. ■

Answer 20 Assuming that the graph is $G = (V, E)$ the problem may be formulated as

$$\begin{array}{lll} \text{minimize} & \sum_{i \in V} x_i & \\ \text{subject to} & x_i + x_j \geq 1 & (i,j) \in E \\ & x_i \in \{0,1\} & i \in V \end{array}$$

Consider a triangle in the graph, e.g. spanned by the nodes $(1,3,6)$. Then we have the inequalities from the formulation

$$\begin{array}{ccccccc} x_1 & + & x_3 & & & \geq & 1 \\ & & x_3 & + & x_6 & \geq & 1 \\ x_1 & & & + & x_6 & \geq & 1 \end{array}$$

using the multipliers $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ for the three inequalities we get the new inequality

$$x_1 + x_3 + x_6 \geq \frac{3}{2}$$

rounding up the right-hand-side gives $x_1 + x_3 + x_6 \geq 2$.

Now, using this inequality for triangles $(1,3,6)$, $(2,4,7)$, $(3,5,1)$, $(4,6,2)$, $(5,7,3)$, $(6,1,4)$ and $(7,2,5)$, using multipliers $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ we get the new inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq \frac{14}{3}$$

rounding up the right-hand-side gives the stated. ■